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SURFACES OF REVOLUTION OF MINIMUM RESISTANCE.

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THE problem of finding the surface of revolution of minimum resistance may be thought of as the oldest problem of the calculus of variations. A first solution was given by Newton* in 1686. It has since been considered by L'Hospital, August, Silvabelle, Kneser, and others. The results obtained by these writers are based on the Newtonian law of resistance, which states that the resistance R is given by the formula

$$R = f. \sin^2 \alpha,$$

where f is the force and α the angle which the line of force makes with the tangent to the surface at the point of application. However, physical experiment does not always verify this law. Especially does it fail† when the angle α is small.

As a result of this, several different laws of resistance have been given; some being derived mathematically, others being stated as verifying experiment. Among these the laws of von Lössl,‡ Duchemin,§ and Kirchhoff|| have received the greater notice. They are given by the following formulas:

$$R = f \sin \alpha, \quad R = f \frac{2 \sin \alpha}{1 + \sin^2 \alpha}, \quad R = f \frac{(4 + \pi) \sin \alpha}{4 + \pi \sin \alpha}.$$

* See Principia Philosophiæ Naturalis, II; Sect. VII, Prop. XXXIV, Scholium.

† For an account of the various physical causes underlying this, see Encyklopädie der mathematischen Wissenschaften, IV, 17, §§ 4, 5, 6.

‡ F. v. Lössl, Die Luftwiderstandsgesetze, Wien, 1896, p. 96.

§ Duchemin, Experimentaluntersuchungen über den Widerstand der Flüssigkeiten, Braunschweig, 1844, p. 101. This law has been verified by Langley, Experiments in Aërodynamics, Washington, 1891, p. 101.

|| G. Kirchhoff, *Journal für Mathematik*, vol. 70 (1869).

The present paper had its origin in a study of the surfaces of revolution resulting from these separate laws. In each case it was found that the curve determining the surface had properties quite analogous to those of the newtonian curve. This naturally suggested the determination of a general law of resistance having properties which would include the preceding as special cases. Such a law is stated in §1. In this section the discussion of the properties of the curves resulting from this law is given. In the following three sections the theory of the special cases will be taken up in the order mentioned as applications of the general. This general case also includes the newtonian law of resistance and from it the well known results* for this law can be obtained.

§ 1. *Surface of Revolution of Minimum Resistance for a General Law of Resistance.*

In formulating the problem it will be supposed that the surface is formed by the revolution of an arc AB

$$x = x(t), \quad y = y(t) \qquad (t_1 \leqq t \leqq t_2),$$

about the X-axis, where the point A is taken at the origin and B in the interior of the first quadrant. The direction cosines of the tangent are then given by the expressions $x'(t)$ and $y'(t)$. Let this arc be such that $y > 0$ except for $t = t_1$. Further it will be supposed that the body moves in the direction of the negative X-axis with constant velocity.

Consider then the surface resulting from the law of resistance expressed by the formula

$$R = f \cdot \varphi(x', y'), \tag{1}$$

where the function $\varphi(x', y')$ is homogeneous and of dimension zero in x' and y', and f again represents the force. It is readily shown that, aside from a numerical factor, which is independent of the form of the curve, the resistance is given by the definite integral

$$J = \int_{t_1}^{t_2} yy'\varphi(x', y')dt. \tag{2}$$

However from physical considerations† it is necessary to limit

* For a very complete summary of the work done on this classical problem see Bolza, Vorlesungen über Variationsrechung, pp. 407–418.

† See August, *Journal für Mathematik*, vol. 103 (1888), p. 1.

the discussion to curves such that along the arcs AB one has

$$x' \geqq 0, \quad y' \geqq 0.$$

Consequently the problem under consideration may be stated thus:

Among all ordinary curves which go from the origin A to a point B given in the interior of the first quadrant and which besides satisfying the regional restriction

$$(3) \qquad y > 0 \quad \text{for} \quad t_1 < t \leqq t_2$$

also satisfy the slope condition

$$(4) \qquad x' \geqq 0, \quad y' \geqq 0 \quad \text{for} \quad t_1 \leqq t \leqq t_2,$$

it is required to find that one which minimizes the integral (2).

Since the restriction has been made that the expression $\varphi(x', y')$ is homogeneous and of dimension zero it follows that it can be written as a function of q, say $\varphi(q)$ for simplicity, where

$$(5) \qquad q = x'/y' = \cot \theta,$$

θ being the angle which the tangent makes with the positive X-axis.

The following restriction will now be imposed on the function $\varphi(q)$:

Its derivative $\varphi'(q)$ first decreases continuously from zero to a finite minimum value and then constantly increases to the value zero as q increases from 0 *to* $+\infty$.

Graphically the function $\varphi'(q)$ is as shown in Fig. 1. If c denotes the value of q, when $\varphi'(q)$ has its minimum, then $\varphi'(q)$ assumes all values between 0 and $\varphi'(c)$ twice.

These conditions are fulfilled in all the special cases to be considered.

Suppose now that an arc of the minimizing curve is considered which is of class C' and such that

$$(6) \qquad x' > 0, \quad y' > 0.$$

This arc must then satisfy the Euler differential equation, from which it follows that a first integral is given by the equation

$$(7) \qquad yy'\partial\varphi(x', y')/\partial x' \equiv y\varphi'(q) = -a,$$

where a is a constant of integration which must be positive on account of the above conditions. Hence it follows that

$$y = -\frac{a}{\varphi'(q)} = aY(q).$$

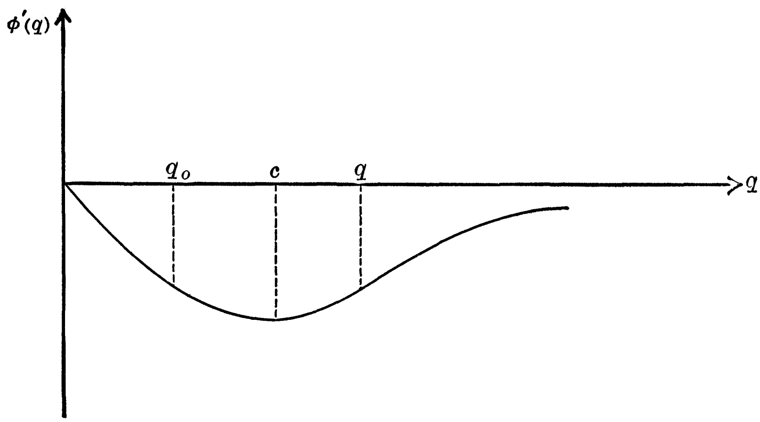

FIG. 1.

But from the relation (5) x' is seen to have the value

$$x' = a\frac{\varphi''(q)}{\varphi'^2(q)}qq'.$$

and therefore x is of the form

$$x = aX(q) + b, \quad X(q) = \int \frac{\varphi''(q)}{\varphi'^2(q)}qdq.$$

The two equations

$$(8) \qquad x = aX(q) + b, \quad y = aY(q),$$

furnish the most general solution of the Euler differential equations in terms of the parameter q, when the inequality (6) is satisfied.

But in as much as the general extremal can be obtained from the special curve

$$(9) \qquad X = X(q) \equiv \int \frac{\varphi''(q)}{\varphi'^2(q)}qdq, \quad Y = Y(q) \equiv \frac{-1}{\varphi'(q)},$$

by means of the similarity transformation

(10) $$x = aX + b, \quad y = aY,$$

it is only necessary to study the special curve (9) in order to find the general properties of the extremal curve.

In order to discuss the curve (9) the derivatives will be computed and the expression for the curvature determined under the assumption that both X and Y are infinite for $q = +0$ and $q = +\infty$. It is readily verified that

$$X' = \frac{\varphi''}{\varphi'^2} q, \qquad Y' = \frac{\varphi''}{\varphi'^2},$$

(10) $$X'' = \frac{\varphi'\varphi'' + q\varphi'\varphi''' - 2q\varphi''^2}{\varphi'^3}, \qquad Y'' = \frac{\varphi'\varphi''' - 2\varphi''^2}{\varphi'^3},$$

$$X'Y'' - Y'X'' = \frac{\varphi''^2(q)}{\varphi'^4(q)}.$$

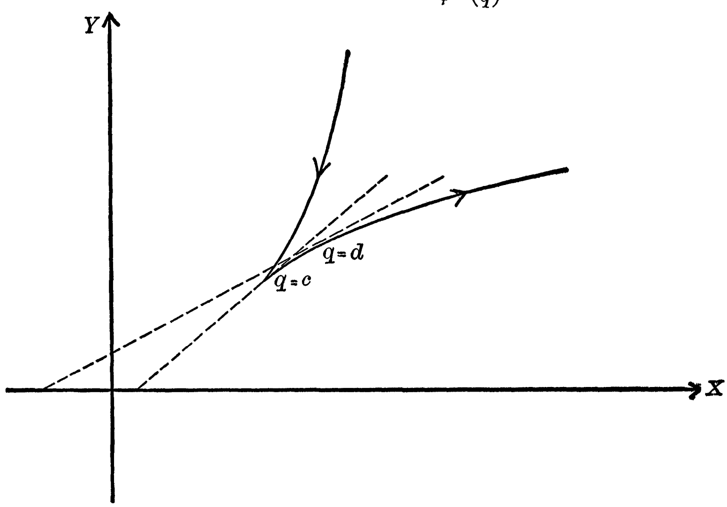

FIG. 2.

Hence the curve has a cusp when $\varphi''(q) = 0$. Let the corresponding value of q be $q = c$. Further, for the range of values

$$0 \leqq q \leqq c$$

the curve has an infinite branch which is convex towards the X-axis, and for

$$c \leqq q < +\infty$$

an infinite branch concave towards the X-axis. The curve is then as shown in Fig. 2.

However the Legendre condition, $F_1 \geqq 0$, shows that the following inequality must hold

$$F_1 \equiv \frac{y}{y'^3}\varphi''(q) \geqq 0.$$

This excludes the possibility of the convex arch of the curve ever furnishing a minimum and thus leaves only the concave arch to be considered.

Further consideration shows that in general the Weierstrass condition excludes a part of this arch. For recalling the definition of the Weierstrass E-function, viz.,

$$E(x, y;\ x', y';\ \bar{x}', \bar{y}') = F(x, y, \bar{x}', \bar{y}') - [\bar{x}'F_{x'}(x, y, x', y') + \bar{y}'F_{y'}(x, y, x', y')]$$

it is seen that in this case

$$\begin{aligned} E(x, y;\ x', y';\ \bar{x}', \bar{y}') &= y\bar{y}'\left\{\varphi(\bar{x}', \bar{y}') - \varphi(x', y') - \frac{\bar{x}'}{\bar{y}'}y'\varphi_{x'}(x', y') - y'\varphi_{y'}(x', y')\right\} \\ &= y\bar{y}'\{\varphi(\bar{q}) - \varphi(q) - [\bar{q} - q]\varphi'(q)\}. \end{aligned}$$

Since however y and $\bar{y}'$ are both positive, it is seen that the E-function will be positive under the following condition:

$$(11) \qquad \Phi(\bar{q}) \equiv \varphi(\bar{q}) - \varphi(q) - [\bar{q} - q]\varphi'(q) \geqq 0,$$

for $c \leqq \bar{q} < +\infty$ and $0 \leqq \bar{q} < +\infty$.

Consider now the derivative of $\Phi(q)$. This is

$$\Phi'(\bar{q}) = \varphi'(\bar{q}) - \varphi'(q).$$

From the graph of $\varphi'(q)$ it follows that the second term of the above derivative is always positive, while $\varphi'(\bar{q})$ is always negative, decreasing from zero to a finite negative value when $\bar{q} = c$ and then increasing to 0 as $\bar{q}$ approaches infinity. Hence for any fixed value of q in the range $c \leqq q < +\infty$ the derivative $\Phi'(q)$ is first positive in an interval $0 \leqq \bar{q} < q_0$, where q_0 is the value of $\bar{q}$ which makes $\varphi'(\bar{q}) = \varphi'(q)$. Then in the interval $q_0 < \bar{q} \leqq q$ the derivative is negative. Finally for $q \leqq \bar{q} < +\infty$ the derivative is again positive. Hence in these intervals the function $\Phi(\bar{q})$ first increases, then decreases to the value 0 and again increases, as shown in Fig. 3.

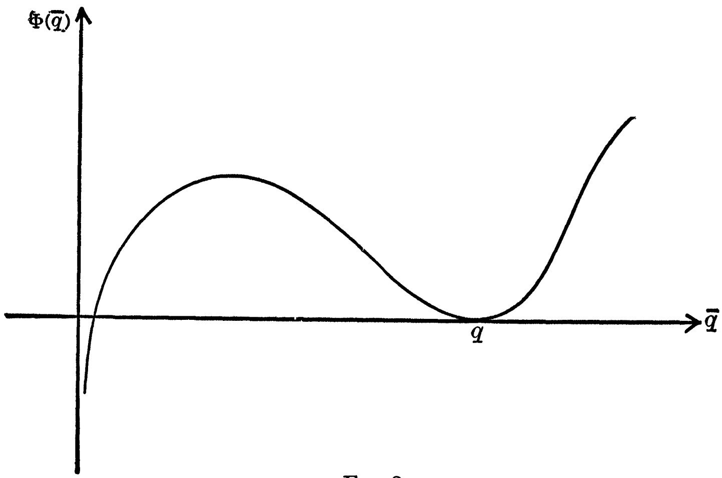

FIG. 3.

Therefore in order that $\Phi(\bar{q})$ may always be positive *it is necessary that*

(12) $\varphi(0) - \varphi(q) + q\varphi'(q) \geqq 0$ for $c \leqq q < +\infty$;

i. e., *if* $\varphi(0) - \varphi(c) + c\varphi'(c) \geqq 0$, *then the Weierstrass condition is satisfied at each point of the concave arch of the curve and a strong minimum results. If however*

(13) $$\varphi(0) - \varphi(c) + c\varphi'(c) < 0,$$

then the equation

(14) $$\varphi(0) - \varphi(q) + q\varphi'(q) = 0$$

has one root $d > c$ *and the Weierstrass condition requires that* $q \geqq d$. Only that part of the concave arch beyond the point where $q = d$ furnishes a strong minimum.

§ 2. *The von Lössl Surface.*

In case the law of resistance is taken as that of von Lössl equation (1) becomes

$$R = f \cdot \sin\theta,$$

and the curves defining the surface of revolution must be found among those which minimize the definite integral

$$J = \int_{t_1}^{t_2} y \frac{y'^2}{\sqrt{x'^2 + y'^2}}\, dt. \tag{15}$$

A first integral of Euler's equation follows at once and the equations of the extremal curves in terms of the parameter q are

$$(16)\qquad \begin{aligned} x &= a\left[\tfrac{2}{3}(q^2+1)^{\frac{3}{2}} - (q^2+1)^{\frac{1}{2}} + \log\frac{1+\sqrt{1+q^2}}{q}\right] + b, \\ y &= a\frac{(q^2+1)^{\frac{3}{2}}}{q}. \end{aligned}$$

In order to find the value of q for which the extremal has a cusp it is necessary to solve the equation

$$\varphi''(q) = 0$$

for q, where

$$\varphi(q) = \frac{1}{\sqrt{q^2+1}}.$$

This condition requires that $2q^2 - 1 = 0$. So for this problem $c = 1/\sqrt{2}$ and therefore the angle which the cusp tangent makes with the X-axis is

$$\theta = 54^\circ\, 44'.$$

But not all of the concave arch furnishes a minimum, for it can be verified that the inequality (13) is satisfied. Hence in order to find where the minimum actually begins it is necessary to solve the equation resulting from (14), viz.,

$$(q^2+1)^{\frac{3}{2}} - 2(q^2+1) + 1 = 0. \tag{17}$$

This is a cubic of the form

$$x^3 - 2x + 1 \equiv (x-1)(x^2 - x - 1) = 0,$$

where $(q^2+1)^{\frac{1}{2}}$ has been replaced by x. The value $x = 1$ is at once excluded since this would mean that q had the value zero. Solving the quadratic factor for x and taking the positive root, it is at once verified after substituting that q has the value $q = 1.2719 + \equiv d$ and the corresponding value of θ is

$$\theta = 38^\circ\ 10' +.$$

Hence *beyond the point where θ has this value the von Lösslian curve furnishes a strong minimum.*

Further consideration shows* that if the inequalities (6) are not satisfied on an arc of the minimizing curve, then this arc is a segment of the straight line $x =$ constant or $y =$ constant. Consequently, if a minimizing curve exists it must be composed of a finite number of combinations of von Lösslian curves and segments of these straight lines. But it is found on applying the corner condition for discontinuous solutions that the only combination which satisfies all the conditions is the one where a portion of the y-axis is followed by a von Lösslian curve.

Finally, by a method similar to that given by Kneser† for the newtonian problem it can be shown that *through each point P_2 in the interior of the first quadrant there passes one and but one von Lösslian curve which makes the angle* $38^\circ\ 10' +$ *with the positive x-axis at its initial point in the positive y-axis.*

§ 3. *The Duchemin Surface.*

For Duchemin's law of resistance,

$$R = f \cdot \frac{2 \cdot \sin\theta}{1 + \sin^2\theta},$$

aside from a constant factor, the resistance integral is

$$J = \int_{t_1}^{t_2} yy'^2 \frac{(x'^2 + y'^2)^{\frac{1}{2}}}{x'^2 + 2y'^2}\, dt.$$

Expressed in terms of the parameter q, the minimizing curve is then found to have the equations

$$(18)\qquad \begin{aligned} x &= a\left[\tfrac{1}{3}(2q^2-1)(q^2+1)^{\frac{1}{2}} + \frac{6(q^2+1)^{\frac{1}{2}}}{q^2} + 6\log\frac{1+\sqrt{1+q^2}}{q}\right] + b,\\ y &= a\frac{(q^2+2)^2(q^2+1)^{\frac{1}{2}}}{q^3}. \end{aligned}$$

* This statement can be proved by a discussion quite analogous to that found in Bolza, l. c., p. 412.

† *Archiv der Mathematik und Physik*, ser. 3, vol. 2 (1902), p. 273.

Since in this case

$$\varphi(q) = \frac{\sqrt{q^2+1}}{q^2+2}$$

the equation which gives the cusp is

$$(19) \qquad 2q^4 - 3q^2 - 6 = 0.$$

From this it is seen that

$$q^2 = \frac{3+\sqrt{57}}{4},$$

and therefore the angle which the cuspidal tangent makes with the X-axis is

$$\theta = 31^\circ\ 37'.$$

Again it is at once verified that not all of the concave arch furnishes a minimum. In order to find d it is necessary to solve the equation

$$(20) \qquad \frac{1}{2} - \frac{\sqrt{q^2+1}}{q^2+2} - \frac{q^4}{(q^2+2)^2\sqrt{q^2+1}} = 0.$$

After reduction by the substitution of x for $\sqrt{q^2+1}$, equation (20) assumes the form

$$(21) \qquad (x-1)^2(x^3 - 2x^2 - 3x - 2) = 0.$$

The value $x = 1$ is again excluded and the positive root of the cubic is found to be

$$x = 3.1515,$$

from which

$$q = 2.9887 \equiv d$$

and therefore

$$\theta = 18^\circ\ 30'.$$

Hence the concave arch of the curve beyond the point where $\theta = 18^\circ\ 30'$ furnishes a minimum.

As in the first example, the most general solution is found to consist of a portion of the y-axis followed by one of the above curves. Furthermore there is always a unique determination of the constants for such curves when only the part of the curve furnishing a strong minimum is considered.

§ 4. *The Kirchhoff Surface.*

According to Kirchhoff, the law of resistance should be

$$R = f \cdot \frac{(4 + \pi) \sin \alpha}{4 + \pi \sin \alpha}.$$

Neglecting a constant factor which does not affect the shape of the minimizing curve, the resistance integral is found to be

$$J = \int_{t_1}^{t_2} y \frac{y'^2}{\sqrt{x'^2 + y'^2} + ey'} dt,$$

where $e = \frac{1}{4}\pi$.

The equations of the minimizing curve are then found to be

$$(22) \quad \begin{aligned} x &= a\left[- \tfrac{4}{3}(q^2 + 1)^{\frac{3}{2}} + (1 + 2q^2)(q^2 + 1)^{\frac{1}{2}} + eq^2 \right. \\ &\qquad \left. + (1 + e^2) \log \frac{1 + \sqrt{1 + q^2}}{q} - 2e \log q \right] + b, \\ y &= a \frac{\sqrt{1 + q^2}(\sqrt{1 + q^2} + e)^2}{q}. \end{aligned}$$

From the value of $\varphi(q)$ it is found that the equation defining the cusp value of q is

$$(23) \quad 2x^3 - 3x - .7854 = 0,$$

where again $\sqrt{1 + q^2}$ has been replaced by x. The positive root of this equation is found to be

$$x = 1.34603$$

and hence $q = c = .90098$. Therefore at the cusp the tangent makes an angle

$$\theta = 47° \; 59'$$

with the positive X-axis.

However the inequality (13) is again satisfied and so the root d of equation (14) must be found. For this problem the equation (14) becomes

$$x^3 - 2x^2 - ex + 1 + e \equiv (x - 1)\{x^2 - x - (1 + e)\} = 0,$$

where x has the same meaning as above. The root $x = 1$ does not satisfy the previous conditions. The positive root of the quadratic factor is

$$x = 1.9266.$$

From this

$$q \equiv d = 1.64676$$

and therefore

$$\theta = 31^\circ\ 16'.$$

Hence *the concave arch of the curve* (22) *furnishes a minimum for the integral J beyond the point where the tangent makes an angle of* 31° 16′ *with the positive x-axis.*

A statement similar to those in the two preceding sections regarding the most general solution and the determination of the constants holds for this problem.

It should be noted that in neither of the three cases considered is the angle θ corresponding to $q = d$ as large as that of the newtonian problem where $\theta = 45^\circ$. So whether or not the newtonian law fails for small values of the angle α, it is certain that these laws hold *only* for smaller values of the angle than in the newtonian problem.

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SHORTER NOTICES.

Les Systèmes d'Équations aux Dérivées partielles. By CHARLES RIQUIER. Paris, Gauthier-Villars, 1910. xxvii + 590 pp.

DURING the past twenty years Professor Riquier has published a large number of memoirs on the theory of systems of partial differential equations. The main results of his investigations are now made more accessible to mathematicians by incorporating them in a systematic treatise where they are presented from a uniform point of view. The theory of the most general system, containing any number of equations involving any number of functions of any number of independent variables with their partial derivatives of arbitrary order—is naturally extremely difficult, and the author is to be congratulated for the clearness of his treatment. The symbolism and terminology are carefully chosen, the main