DEFINITE INTEGRALS CONTAINING A PARAMETER.

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A FUNCTION $f(\alpha, x)$ is defined for each pair of values of α and x in the closed region $0 \leqq \alpha \leqq 1$ and $0 \leqq x \leqq 1$. For each value of α in the interval $(0, 1)$ the function $f(\alpha, x)$ is an integrable function of x according to Riemann's definition. A function $F(\alpha)$ is thus defined by the equation

$$F(\alpha) = \int_0^1 f(\alpha, x)dx.$$

The problem considered in this paper is one of uniform convergence; namely, the determination of the conditions to be imposed on the function $f(\alpha, x)$ in order that corresponding to any positive number ϵ there exist a number δ independent of α such that

$$\text{(I)} \qquad \left| F(\alpha) - \sum_{i=0}^{i=n} f(\alpha, \xi_i)(x_i - x_{i-1}) \right| < \epsilon,$$

$$(x_0 = 0, \quad x_n = 1, \quad x_{i-1} \leqq \xi_i \leqq x_i)$$

for $(x_i - x_{i-1}) < \delta$.

Closely associated with this problem of uniform convergence are, at any rate, two others which lend interest to it. Of these, one is the problem concerning the continuity of $F(\alpha)$. Under the assumption that $f(\alpha, x)$ is a continuous function of α for each value of x, a necessary and sufficient condition that $F(\alpha)$ be a continuous function of α follows from the theory developed. The conditions under which the roots of the equation $F(\alpha) = 0$ are limiting points of the roots of the sequence of equations

$$\sum_{i=0}^{i=n} f(\alpha, \xi_i)(x_i - x_{i-1}) = 0$$

as n becomes infinite is the second problem.

The absence of continuity conditions does not preclude the existence of the inequality (I).

Example (a):

$$f(\alpha, x) = 1 + 2x \text{ for } \alpha \text{ rational,}$$

$$f(\alpha, x) = 0 \text{ for } \alpha \text{ irrational,}$$

$$F(\alpha) = 2 \text{ for } \alpha \text{ rational; } F(\alpha) = 0 \text{ for } \alpha \text{ irrational.}$$

For a fixed value of x, $f(\alpha, x)$ is discontinuous in α at every point of the interval (0, 1), and $F(\alpha)$ is also discontinuous at every point. Inequality (I) nevertheless exists.

On the other hand it is not sufficient for the existence of the inequality (I) that $f(\alpha, x)$ be limited and $F(\alpha)$ be continuous.

Example (b):

$$f(\alpha, x) = \sin\frac{2\pi x}{\alpha} \text{ for } \alpha \neq 0, \quad f(\alpha, x) = 0 \text{ for } a = 0,$$

$$F(\alpha) = \int_0^1 \sin\frac{2\pi x}{\alpha}\, dx = \frac{\alpha}{2\pi} - \frac{\alpha}{2\pi}\cos\frac{2\pi}{\alpha} \text{ for } \alpha \neq 0,$$

$$F(\alpha) = 0 \text{ for } \alpha = 0.$$

F(α) is therefore a continuous function of α in the interval (0, 1). Let the law of subdivision be such that the end points of the νth subdivisions coincide with those values of x for which $\sin 2\nu\pi x$ has maximum values, and let $\xi_i = x_i$. There will then exist an integer n greater than any fixed integer m, and a positive number α less than any fixed positive number δ, such that

$$\left|\sum_{i=1}^{i=n} f(\alpha, \xi_i)(x_i - x_{i-1}) - 1\right| < \epsilon,$$

where ϵ is any preassigned positive number. Since, however,

$$\lim_{\alpha=0} F(\alpha) = 0,$$

the inequality (I) does not exist.

Nor, again, is it sufficient for the existence of the inequality that both $F(\alpha)$ be continuous in α and $f(\alpha, x)$ be a continuous function of α for each value of x.

Example (c):

$$f(\alpha, x) = \frac{\alpha^{\frac{3}{2}}}{\alpha^2 + (x-1)^2} \text{ for } x \neq 1,$$

$$f(\alpha, x) = 0 \text{ for } x = 1.$$

$f(\alpha, x)$ is then a continuous function of α for each value of x. The function

$$F(\alpha) = \int_0^1 f(\alpha, x)dx = \alpha^{\frac{1}{2}} \operatorname{arctg} \frac{1}{\alpha} \text{ for } \alpha \neq 0,$$

$$= 0 \text{ for } \alpha = 0$$

is continuous in α throughout the interval (0, 1). Let the end points of the νth subdivision fall at

$$0, \quad \frac{1}{\nu}, \quad \frac{2}{\nu}, \quad \frac{3}{\nu}, \quad \frac{4}{\nu}, \quad \cdots, \quad \frac{\nu - 1}{\nu}, \quad 1$$

and let $\xi_i = x_i$ for $i \neq \nu$ and $\xi_\nu = 1 - 1/\nu^2$; then the sum having $F(\alpha)$ for its limits is

$$\frac{1}{\nu} \cdot \frac{\alpha^{\frac{3}{2}}}{\alpha^2 + \left(\frac{1}{\nu} - 1\right)^2} + \frac{1}{\nu} \frac{\alpha^{\frac{3}{2}}}{\alpha^2 + \left(\frac{2}{\nu} - 1\right)^2} + \cdots + \frac{1}{\nu} \cdot \frac{\alpha^{\frac{3}{2}}}{\alpha^2 + \left(1 - \frac{1}{\nu^2} - 1\right)^2}.$$

For $\alpha = 1/\nu^2$ the last term of the sum is $\frac{1}{2}$; the sum of all the terms is greater than $\frac{1}{2}$ since all the terms are positive. The function $F(\alpha)$ is continuous and equals zero when α equals zero, the inequality (I) therefore does not exist.

THEOREM I. *If $f(\alpha, x)$ be a continuous function of the two variables α and x, the inequality* (I) *exists.*

This theorem is included in the following more general

THEOREM II. *If $f(\alpha, x)$ be continuous in α uniformly with respect to x, i. e., if corresponding to any positive number ϵ there exists a δ independent of x such that $|f(\alpha + h, x) - f(\alpha, x)| < \epsilon$ for $|h| < \delta$, the inequality* (I) *exists.*

This theorem again is a special case of theorem (III).

Theorems I and II are stated under the assumption that $f(\alpha, x)$ is an integrable function of x for each value of α. In Theorem III we drop this requirement and assume only that $f(\alpha, x)$ is a limited function of x for each α.

THEOREM III. *If $f(\alpha, x)$ be continuous in α uniformly with respect to x, then the upper sum* converges to the upper integral* uniformly with respect to α.*†

Since $f(\alpha, x)$ is a limited function of x for each α, the upper integral exists and a function $\bar{F}(\alpha)$ is defined by the equation

$$\bar{F}(\alpha) = \bar{\int_0^1} f(\alpha, x)dx.$$

The upper limit of $f(\alpha, x)$ for values of x in the interval (x_i, x_{i-1}) is denoted by $\bar{f}_i(\alpha)$. Unless the upper sum converges to the upper integral uniformly in α throughout the interval $(0, 1)$, there exists a point $\alpha = b$ in this interval such that in any arbitrarily small interval about b the convergence is not uniform. It is sufficient then to establish uniform convergence in the neighborhood of b.

$$\left| \bar{F}(b + \eta) - \sum_{i=1}^{i=n} \bar{f}_i(b + \eta)(x_i - x_{i-1}) \right|$$
$$\leqq \left| \sum_{i=1}^{i=n} \bar{f}_i(b)(x_i - x_{i-1}) - \bar{f}_i(b + \eta)(x_i - x_{i-1}) \right|$$
$$+ \left| \bar{F}(b) - \Sigma \bar{f}_i(b)(x_i - x_{i-1}) \right| + \left| \bar{F}(b + \eta) - \bar{F}(b) \right|.$$

Corresponding to any positive number ϵ, there exists a number δ_1 independent of i such that

$$|\bar{f}_i(b) - \bar{f}_i(b + \eta)| < \frac{\epsilon}{3}$$

for $|\eta| < \delta_1$. This follows from the assumption that the function $f(\alpha, x)$ is continuous in α uniformly with respect to x. The existence of the upper integral

$$\bar{\int_0^1} f(b, x)dx$$

renders it possible to choose a number δ_2 such that

$$\left| \bar{F}(b) - \sum_{i=1}^{i=n} \bar{f}(b)(x_i - x_{i-1}) \right| < \frac{\epsilon}{3} \quad \text{for } (x_i - x_{i-1}) < \delta_2$$

* Cp. Pierpont's Functions of a Real Variable, vol. 1, p. 337; Hobson's Functions of a Real Variable, p. 339.

† The same theorem holds, of course, for the lower sum and lower integral.

$$|\bar{F}(b+\eta)-\bar{F}(b)| = \left|\bar{\int_0^1} f(b+\eta, x)dx - \bar{\int_0^1} f(b, x)dx\right| \leqq \left|\bar{\int_0^1} \{f(b+\eta, x)-f(b, x)\}\right| dx$$

$$\leqq \text{the upper limit of } |f(b+\eta, x)-f(b, x)|.$$

Hence there exists a number δ_3 such that

$$|\bar{F}(b+\eta)-\bar{F}(b)| < \frac{\epsilon}{3} \quad \text{for} \quad |\eta| < \delta_3.$$

Thus we have established the existence of a number δ such that

$$\left|F(b+\eta) - \sum_{i=1}^{i=n} \bar{f}_i(b+\eta)(x_i - x_{i-1})\right| < \epsilon \text{ for } |\eta| < \delta$$

and $(x_i - x_{i-1}) < \delta$.

THEOREM IV. *If the function* $F(\alpha)$ *be continuous in the interval* $(0, 1)$, *and the function* $f(\alpha, x)$ *be a continuous function of* α *for each* x, *then under any fixed law of subdivision there will correspond to any integer* m *and any positive number* ϵ *an integer* $n > m$ *and a number* δ *such that*

$$\left|F(b+\eta) - \sum_{i=1}^{i=n} f(b+\eta, x_i)(x_i - x_{i-1})^*\right| < \epsilon \text{ for } |\eta| < \delta.$$

THEOREM V. *If* $f(\alpha, x)$ *be a continuous function of* α *for each* x, $F(\alpha)$ *will be continuous at* b *provided that, corresponding to any positive number* ϵ *and any integer* m, *there exist a number* δ *and an integer* $n > m$ *such that*

$$| F(b+\eta) - \sum_{i=1}^{i=n} f(b+\eta, x_i)(x_i - x_{i-1})^* | < \epsilon \text{ for } |\eta| < \delta.$$

The proofs of Theorems IV and V are almost identical with the proofs of the two theorems which establish the necessary and sufficient condition that the sum of an infinite series of continuous functions shall be a continuous function. These theorems in infinite series, as well as those stated here for definite integrals containing a parameter, are applications of

* In the function $f(b+\eta, x_i)$ x_i may be replaced by ξ_i provided the manner of assigning ξ_i is prescribed.

the following theorem concerning functions defined by sequences of continuous functions. We assume that each of the sequence of functions $\varphi_n(x)$ is continuous in the interval (0, 1), and that $\lim_{n=\infty} \varphi_n(x) = \varphi(x)$ exists. A necessary and sufficient condition for the continuity of $\varphi(x)$ in the interval (0, 1) is that, corresponding to any positive number ϵ and any integer m, the condition $| \varphi(x) - \varphi_n(x) | < \epsilon$ is satisfied for every value of x in (0, 1), where n has one of a finite number of values all greater than m, the value to be given to n depending on the value assigned to x.

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ON THE V_3^3 WITH FIVE NODES OF THE SECOND SPECIES IN S_4.

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CUBIC varieties in four-space were first investigated by Segre, in two memoirs* which are still classic, and in which he gave a generation of those having more than six nodes, especially the one with ten nodes, while he also considered varieties containing a plane, and gave some of their properties. Castelnuovo† investigated also the V_3^3 with ten nodes, and a good account of the theory of the latter is to be found in Bertini.‡ So far as we know however, varieties having nodes for which the hypercone tangent degenerates into one cut by any V_3^1 in a cone—points which we define as nodes of the second species—have been but little considered. In a previous paper § the writer has given the maximum of these nodes for surfaces, or rather a method for obtaining it. This method admitted of an evident extension to n-space, and in particular gives for V_3^3 in four-space, a maximum of these nodes equal to half the number of absolute invariants of the most

* "Sulle varietà cubiche," *Memorie dell' Academia di Torino*, ser. 2, vol. 39 (1888). "Sulla varietà cubicha con 10 punti doppi," *Atti di Torino*, vol. 22 (1887).

† "Sulle congruenze dell 3° ordine," *Atti dell' Ist. Veneto*, ser 6, vol. 6 (1888).

‡ Geometria proiettiva degli iperspazi, p. 176.

§ "On the existence of loci with given singularities." Read before the Poughkeepsie meeting of the Society, Sept. 12, 1911.