"limit" may be defined for general systems of objects in an analogous manner, without excluding the possibility of special examples of the Cesàro type.

10. The nodes and perihelia of the four inner planets, notably Venus and Mars, present certain unexplained motions in the Newtonian mechanics. The note of Professor James compares the secular changes in the elements of these planets produced by the uniform rotation of the empirical about the inertial system of reference with the corresponding changes brought about by the use of the Minkowskian law of attraction instead of the Newtonian.

11. In this paper Dr. Frizell shows that a one-to-one relation exists between the continuum and a set of terms in the expansion of an infinite determinant whose elements are restricted to the principal diagonal and two adjacent diagonals.

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SERIES OF LAPLACE'S FUNCTIONS.

BY PROFESSOR B. H. CAMP.

(Read before the American Mathematical Society, October 28, 1911.)

THE most important theorem on the validity of the expansion of an arbitrary function in a series of Laplace's functions has been proved by Jordan in his Cours d'Analyse, second edition, volume 2, page 252. The conditions there stated are that the given function be continuous on the surface of the sphere within some small circle about the point at which the expansion is made, and that it have limited variation along every great circle through this point.

The object of the present paper is to correct an error in Jordan's theorem, and to furnish new conditions sufficient for the validity of these expansions. To the conditions announced by Jordan should be added the requirements that the values of the variations be all less than some fixed number, and that these variations be "uniform with respect to all great circles through the point." His error is discussed in a remark following Corollary 2.

In Theorem 1, I have corrected Jordan's theorem and extended it to the case where any or all of the conditions may fail on a null set of great circles through the point. Theorem 2 is a new theorem, and replaces the conditions of continuity of the function and uniformity of the variations, in Theorem 1, by the different requirement that there exist a small circle about the point considered such that within it the given function is, along every great circle through the point, an indefinite integral of another function which has an absolutely convergent double Lebesgue integral in its domain of definition. A null set of great circles may be neglected in this theorem also.

All the integrals used are L-integrals, i. e., integrals in the sense of Lebesgue. Limited variation may be understood to refer either to the definition of Jordan or to that of Pierpont; the theorems are true for both definitions.

Suppose the function $f(p)$ to be defined and limited and to have a double Lebesgue integral on the surface of a unit sphere, p being a point of the surface, and suppose it to be required to develop this function in a series of Laplace's functions which shall be valid at a fixed point p_0. We learn from Jordan's work, subject to certain transformations which are permissible* in the present case, that the validity of the formal development depends on the convergence to $\Phi(x_0)$ of

$$(1) \qquad S_n = \frac{1}{4\pi}\int_0^{2\pi} d\mu \int_{-1}^{+1} \Phi(x, \mu)\, [X'_{n+1}(x) + X'_n(x)]dx,$$

where $\Phi(x, \mu)$ is the form assumed by $f(p)$ when the coordinates (x, μ) of p are chosen as stated below, and X'_1 is the derivative with respect to x of X_n, the function of Legendre. Here p_0 is taken as the north pole of the sphere, x as the cosine of the colatitude of p, and μ as the longitude of p with respect to some fixed meridian $\mu = 0$. Evidently $f(p_0) = \Phi(x_0) = \Phi(1)$, and is independent of μ. Now since

$$\int_{-1}^{+1} (X'_{n+1} + X'_n)\, dx = 2,$$

it follows that

$$\Phi(x_0) = \frac{1}{4\pi}\int_0^{2\pi} d\mu \int_{-1}^{+1} \Phi(x_0)\, (X'_{n+1} + X'_n)dx.$$

* Lebesgue, *Annales l'Ecole Norm. Sup.*, ser. 3, vol. 27 (1910), pp. 447–50.

Subtracting this from (1), we have, placing $\psi(x, \mu) = \Phi(x, \mu) - \Phi(x_0)$,

$$(2) \quad 4\pi(S_n - \Phi(x_0)) = \int_0^{2\pi} d\mu \int_{-1}^{+1} \psi(x, \mu)\, (X'_{n+1} + X'_n) dx,$$

and the validity of the expansion to be studied depends on the convergence of (2) to zero with $1/n$.

LEMMA 1. *If $f(x)$ has limited variation of value V in the interval (a, b), and $\phi(x)$ is absolutely L-integrable in this interval,*

$$\left| \int_a^b f\phi \right| \leq (V + \max |f|) \max_{\lambda} \left| \int_a^{\lambda} \phi \right|, \; \lambda \text{ in } (a, b).$$

This is readily deduced from a theorem of Lebesgue's (*Annales de la Faculté de Toulouse,* series 3, volume 1 (1909), page 37).

THEOREM 1. (1°) *Let $\Phi(x, \mu)$ be defined and limited and have a double* L-integral in the rectangle $R = (-1 \leq x \leq 1, 0 \leq \mu \leq 2\pi)$;* (2°) *let it be, at $x = 1$, a function of x uniformly continuous with respect to μ;* (3°) *let it have limited variation in $(-1, 1)$, and let the value of this variation be limited with respect to μ.* (4°) *Let the variation of Φ in $(1 - x, 1)$ be, at $x = 1$, a function of x uniformly continuous with respect to μ. Finally, any or all of these conditions may fail for a null of set μ's.*

Then the development of $f(p)$ in Laplace's functions is valid at p_0.

Proof. We will establish the theorem by showing that (2) converges to zero with $1/n$. Let ϵ be > 0 and arbitrary, and select h so that by 2°

$$(3) \quad \max |\psi| < \epsilon \text{ in } (1 - h, 1)$$

for all μ's uniformly, except perhaps a null set; and so that by 4°, for the function ψ,

$$(4) \quad V(1 - h, 1) < \epsilon$$

for all μ's uniformly, except perhaps a null set. We may

* It is known that

$$\int_R \psi(X'_{n+1} + X'_n) = \int_0^{2\pi} d\mu \int_{-1}^{+1} \psi(X'_{n+1} + X'_n) dx.$$

A slightly more general condition would suffice, but the difference is trivial.

write (2) in the form

$$
(5)\quad \begin{aligned}&\int_0^{2\pi} d\mu \int_{-1}^{1-h} \psi(X'_{n+1} + X'_n)dx \\ &+ \int_0^{2\pi} d\mu \int_{1-h}^{1} \psi(X'_{n+1}+X'_n)dx = \int_0^{2\pi} (I)\, d\mu + \int_0 \quad (II) d\mu.\end{aligned}
$$

By 3° and Lemma 1,

$$
|\,\mathrm{II}\,| \leq (V(1-h, 1) + \max |\,\psi\,|\,) \max_{\lambda} \left| \int_{\lambda}^{1} (X'_{n+1} + X'_n)dx \right|,
$$

λ in $(1-h, 1)$, for all μ's, except perhaps a null set. But

$$
\left| \int_{\lambda}^{1} (X'_{n+1} + X'_n)dx \right| = |\, 2 - X_{n+1}(\lambda) - X_n(\lambda) \,| < 4,
$$

and therefore

$$
(6)\qquad |\,\mathrm{II}\,| < 4(\epsilon + \epsilon),
$$

by (3) and (4), for all μ's uniformly, except perhaps a null set. Similarly

$$
|\,I\,| \leq (V(-1, 1-h) + \max |\,\psi\,|\,) \max_{\lambda} \int_{-1}^{\lambda} (X'_{n+1} + X'_n)dx,
$$

λ in $(-1, 1-h)$, for all μ's except perhaps a null set. The integral here equals $X_{n+1}(\lambda) + X_n(\lambda) - 0$, and Jordan has shown (page 236) that this expression converges to zero with $1/n$ uniformly with respect to λ in $(-1, 1-h)$, provided only that $0 < h \leq 2$. Since, by 1° and 3°, there exists an M such that $V + \max |\,\psi\,| < M$, we have now shown that there exists an n_ϵ, independent of μ, so that

$$
(7)\qquad |\,I\,| < M\epsilon, \quad n > n_\epsilon,
$$

for all μ's uniformly except perhaps a null set. This with (6) and (5) shows that

$$
\left| \int_0^{2\pi} d\mu \int_{-1}^{+1} \psi\, (X'_{n+1} + X_n')dx \right| < 2\pi(8\epsilon + M\epsilon), \quad n > n_\epsilon;
$$

which proves the theorem.

REMARK. *In order that* 4° *may be fulfilled, it is sufficient* *that* Φ *be, with respect to* μ, *uniformly continuous at* $x = 1$, *and that in* $(-1, 1)$ *it have limited variation which is "uniform with respect to* μ" *(i. e., that* $lim_{d=0} V_D = V$ *uniformly with respect to* μ, *where* V *is the value of the variation in* $(-1, 1)$, *and* V_D *the usual sum function for the arbitrary division* D *of norm* d*).*

COROLLARY 1. *The same conclusion follows if, in place of* 3° *and* 4°, *we have the condition that* Φ *be a monotone increasing function of* x.

For 3° is then satisfied because $V \leqq \max |\psi|$ in the rectangle R. Moreover, if $d \leqq 2$, $V - V_D = 0$, for all μ's, except perhaps a null set. Therefore, by the Remark above, 4° is also satisfied.

COROLLARY 2. *If* $f(p) = f_1(p) - f_2(p)$, *where* f_1 *and* f_2 *both satisfy the conditions of Corollary* 1, *the same conclusion follows.*

It is this last corollary that Jordan has actually proved, except that he may not neglect a null set of meridians, and that he should have stated that f must be limited on the sphere as well as along each meridian. (Cf. page 230, "D'autre part $\psi(b - 0)$ est fini. . . ." At the bottom of page 251 he desires to have the convergence uniform with respect to μ, and therefore ψ "fini" uniformly with respect to μ.) Having proved this corollary, however, it is not permissible to conclude, as Jordan does, that the theorem will be true without the restrictions stated in 4° and in the last part of 3°. As he remarks in § 220, his theorems are proved for monotone functions, and stated for functions of limited variation. This is a proceeding which is often allowable, because of the well known fact that any function of limited variation is the difference of two monotone increasing functions, but it may lead to error unless the two monotone functions may be made to obey the restrictions imposed on the given function. As a matter of fact, in the case before us, it is not difficult to construct a limited, integrable function, which satisfies the conditions imposed by Jordan, but is not expressible as the difference of two monotone increasing functions each of which separately obeys the same conditions. In a remark appended to a fundamental convergence theorem (§ 222, page 230) Jordan recognizes this difficulty, but he ignores his remark when he afterwards applies his theorem to Laplace's functions. He has actually found, then, the conditions which certain components of the given function should satisfy, not those which are to be satisfied by the function itself.

LEMMA 2. *If of two functions, of one or of two variables, one is absolutely L-integrable in a limited field, and the other is limited, their product is absolutely L-integrable in the same field.**

LEMMA 3. *Let A be any limited, measurable field in the x, y plane. In order that the integral*

$$\int_A f(x, y)\, \phi(x, y;\, n)$$

may converge to zero with $1/n$ for all absolutely L-integrable functions f defined in A, it is necessary and sufficient that there exist an M and an n_M so that $|\phi| < M$, $n > n_M$, except perhaps at a null set of points, and that $\int_r \phi \doteq 0$ with $1/n$ for each rectangle r of sides parallel to the axes x, y.

This may be proved in a manner analogous to the proof given by Lebesgue for the corresponding theorem in one dimension (*Annales de la Faculté de Toulouse,* loc. cit., page 52)† by the use of another theorem proved by him in *Annales de l'Ecole Normale Supérieure,* series 3, volume 27 (1910), page 374.

THEOREM 2. *Using the notation of Theorem* 1, (1°) *let $\Phi(x, \mu)$ be defined, limited, and have a double L-integral in R.* (2°) *In $(1 - h, 1)$, for some $h > 0$ and independent of μ, let Φ be the indefinite integral, with respect to x, of another function $\theta(x, \mu)$, which has an absolutely convergent, double L-integral in its field of definition.* (3°) *Let Φ have limited variation in* $(-1, 1)$ *the value of which is limited with respect to μ. Finally, any or all of these conditions may fail for a null set of μ's.*

Then the development of $f(p)$ in Laplace's functions is valid at p_0.

Proof. As in Theorem 1, we may write (2) in the form (5), and, by virtue of 1° and 3° of our hypothesis, the first of these integrals,

$$(8) \qquad \left| \int_0^{2\pi} d\mu \int_{-1}^{1-h} \psi(X'_{n+1} + X'_n)dx \right| < 2\pi M\epsilon, \quad n > n_\epsilon.$$

* Lebesgue, loc. cit., p. 374.

† Cf. also Hobson, *Proc. London Math. Soc.*, ser. 2, vol. 6 (1908), p. 355.

As for the second, by 2° and a theorem due to Lebesgue,*

$$(9)\qquad \int_{1-h}^{1} \psi(X'_{n+1} + X'_n)dx = [\psi(X_{n+1} + X_n)]_{1-h}^{1}$$

$$-\int_{1-h}^{1}(X_{n+1}+X_n)\theta dx = -\psi(1-h,\ \mu)\cdot(X_{n+1}(1-h)+X_n(1-h))$$

$$-\int_{1-h}^{1}(X_{n+1} + X_n)\theta dx, \quad \text{since } \psi(1,\ \mu) = 0.$$

We may now write the second integral of (5) in the form

$$(10)\ \int_0^{2\pi} d\mu \int_{1-h}^{1} \psi(X'_{n+1} + X'_n)dx = -\int_0^{2\pi} d\mu \int_{1-h}^{1} \psi(1-h,\ \mu)$$

$$\cdot(X_{n+1}(1-h)+X_n(1-h))dx - \int_0^{2\pi} d\mu \int_{1-h}^{1}(X_{n+1}+X_n)\theta dx,$$

provided two of these integrals exist. To show that they do exist let us write R_h for the rectangle $1 - h \leq x \leq 1$, $0 < \mu \leq 2\pi$. Then, by 2° and Lemma 2, $\int_{R_h}(X_{n+1} + X_n)\theta$ exists, and by a recent theorem of Hobson†

$$(11)\qquad \int_{R_h}(X_{n+1} + X_n)\theta = \int_0^{2\pi} d\mu \int_{1-h}^{1}(X_{n+1} + X_n)\theta dx.$$

We knew before that the first integral of (10) exists.

We now employ again that portion of Jordan's work used to derive (7) in Theorem 1, and learn that

$$|\ X_{n+1}(1 - h) + X_n(1 - h)\ | < \epsilon, \quad \text{if } n > n_\epsilon,$$

and by 1° we know that $|\ \psi(1 - h,\ \mu)\ | < M$; and these inequalities hold for all μ's uniformly, except perhaps a null set. By virtue of these and (11), (10) shows us that

$$(12)\ \left|\int_0^{2\pi} d\mu \int_{1-h}^{1} \psi(X'_{n+1} + X'_n)dx\right| < 2\pi\epsilon M + \left|\int_{R_h}(X_{n+1}+X_n)\theta\right|,$$

if $n > n_\epsilon$. Comparing this with (8) we find that in order to establish the theorem it only remains to show that

* Loc. cit., p. 46.

† *Proc. Lond. Math. Soc.*, ser. 2, vol. 8 (1910), p. 30.

$$\int_{R_h}(X_{n+1}+X_n)\theta \doteq 0 \quad \text{with} \quad 1/n. \tag{13}$$

To do this we use Lemma 3, θ being an absolutely L-integrable function by hypothesis, and $|X_{n+1}+X_n|$ being ≤ 2 for all values of n.

Let r be an arbitrary rectangle in R_h whose sides are parallel to the μ, x axes. It only remains to show that

$$\int_r X_{n+1}+X_n) \doteq 0 \quad \text{with} \quad 1/n. \tag{14}$$

By two well known formulas,* if $|k|<1$,

$$\int_k^1 X_n(x)dx = \frac{(1-k^2)(X_n'(k))}{n(n+1)},$$

and

$$X_n'(k) = \frac{nkX_n(k) - nX_{n-1}(k)}{k^2-1}.$$

Therefore

$$\left|\int_k^1 X_n(x)dx\right| < \frac{k+1}{n+1}, \quad \text{since} \quad |X_n(k)|<1.$$

This integral does not depend on μ, and k depends only on r. (14) follows, and the theorem is proved.

The necessary and sufficient conditions that a function be an indefinite integral are given by Lebesgue in the *Rendiconti dell' Accademia dei Lincei,* volume 16 (1907), 1st semester.

The following function satisfies all the conditions of Theorem 2, but not 2° of Theorem 1:

$$\Phi = \frac{1}{\sqrt{\mu}}(x-1) \quad \text{in} \quad 1-\frac{\sqrt{\mu}}{3} \leq x \leq 1, \text{ if } \mu \neq 0$$

$$= -\frac{1}{3} \text{ elsewhere, if } \mu \neq 0$$

$$= 0, \text{ if } \mu = 0.$$

WESLEYAN UNIVERSITY,
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* Cf., e. g., Byerly, Fourier's series and spherical harmonics, pp. 172, 180.