where K'' is a positive constant, and consequently $f(x)/x^r \rightarrow 0$ as $x \rightarrow \infty$, which was what we desired to prove.

In conclusion, we may remark that the theorem may be stated as one of pure integral calculus, without reference to the theory of summability of integrals. Putting $f(x) = \phi(x)x^r$, the theorem thus becomes:

If $\phi(x)$ is uniformly continuous over the infinite interval $x \geqq k > 0$, then the convergence to a limit, as $x \rightarrow \infty$, of the integral

$$\int_0^x \phi(\beta)\beta^r \left(1 - \frac{\beta}{x}\right)^r d\beta$$

requires that $\phi(x)$ shall $\rightarrow 0$ as $x \rightarrow \infty$.

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IRREDUCIBLE HOMOGENEOUS LINEAR GROUPS OF ORDER p^m AND DEGREE p OR p^2.

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(Read before the American Mathematical Society, February 25, 1911.)

No group all of whose non-invariant commutators give invariant commutators besides identity can be simply isomorphic with irreducible groups of different degrees. This category includes all groups of order p^m (p a prime) and classes one, two, and three. Moreover no group of order p^m can be simply isomorphic with irreducible groups of just two different degrees.*

A consideration of these facts gives rise to the query as to whether any group of order p^m can be simply isomorphic with irreducible groups of different degrees, and it is the purpose of this note to answer this question for certain special cases.

In the first place, if G is an irreducible group of order p^m and degree p, it cannot be simply isomorphic with an irreducible group of any other degree, since it contains an abelian subgroup of index † p, and since a group of order p^m with an abelian sub-

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* BULLETIN, vol. 14 (1908), pp. 328, 329.

† *Transactions Amer. Math. Society*, vol. 7 (1906), p. 68. We shall have occasion to make use of the fact, established here, that in an irreducible group of order p^m and degree p, the substitutions commutative with a substitution that gives an invariant commutator besides identity form an abelian subgroup.

group of index p^δ cannot be simply isomorphic with an irreducible group of degree greater than p^δ.*

Suppose that G is an irreducible group of order p^m and degree p^2. If t is a substitution of G that corresponds to an invariant commutator of G' of order p, it is invariant in a subgroup G_1 of order p^{m-1} and G_1 must be reducible since t is invariant in it and is not a similarity substitution. We can so transform G as to exhibit G_1 in a completely reduced form with p irreducible components and with $G \equiv \{G_1, R\}$, where R replaces each variable of any component by the corresponding variable in the succeeding component.†

If G_1 contains a substitution s that has just p conjugates and is commutative with R, s must be the same in each of the components (each of degree p) of G_1. If s_i is that part of s that involves the variables of the ith component of G_1 then s_i is not invariant in its component, since if it were s would be invariant in G_1, and therefore invariant in G. Each component of G_1, as the ith one, being of degree p, contains an abelian subgroup of index p of which s_i is a part, and in the formation of G_1 no substitution of the abelian subgroup in one of these components can be associated with a substitution that is not in the corresponding abelian subgroup of some other component, since otherwise s would have more than p conjugates. Hence G_1 contains an abelian subgroup of index p, and G cannot be simply isomorphic with an irreducible group of any degree other than p^2.

Now t and R must be contained in a subgroup G_2 of order p^{m-1}, since otherwise G' would be of order p^2 and G could not be an irreducible group of degree p^2.‡ If then G_1 contains no substitution with just p conjugates that is commutative with R, G_2 can contain no invariant substitution that is not invariant in G. For every substitution of G_2 is of the form sR^α, where s is a substitution of G_1, and if sR^α is invariant in G_2, we must have $\alpha \equiv 0 \pmod{p}$, since s is commutative with t and R is not. Therefore sR^α is in G_1, and it is commutative with R. But G_1, by supposition, contains no substitution with just p conjugates that is commutative with R.

Hence G_2 is either irreducible or simply isomorphic with each

* Loc. cit., vol. 8 (1907), p. 110.

† But the variables in the last component are not necessarily replaced by the corresponding ones of the first component.

‡ Loc. cit., vol. 7 (1906), p. 67, Theorem II.

of its p irreducible components.* In the latter case, since these components are each of degree p, G_2 contains an abelian subgroup of order p^{m-2} and G cannot be simply isomorphic with an irreducible group of any degree except p^2. In the former case, since G contains an irreducible subgroup of order p^{m-1} and degree p^2, it cannot be simply isomorphic with an irreducible group of any degree except p^2 and p^3, if we assume that no group of order p^{m_1} $(m_1 < m)$ can be simply isomorphic with irreducible groups of degrees p^2 and p^n respectively $(n > 2)$. But no group of order p^m can be simply isomorphic with irreducible groups of just two different degrees. Hence G cannot be simply isomorphic with an irreducible group of any degree except p^2. The assumption upon which this conclusion rests can easily be justified for small values of m. We have therefore proved the following

THEOREM: *An irreducible group of order p^m and degree p^2 (p a prime) cannot be simply isomorphic with an irreducible group of any other degree.*

If we take into consideration the theorem just proved and the facts heretofore cited together with the fact that if G is simply isomorphic with an irreducible group of degree $p^{\frac{1}{2}(m-a)}$, where p^a is the order of the central of G, it cannot be simply isomorphic with an irreducible group of any other degree,† we can readily verify that if a group of order p^m is simply isomorphic with irreducible groups of different degrees, then $m \geqq 12$.

Let G be an irreducible group of order p^m and degree p^2 that does not contain an abelian subgroup of index p^2. Then G_1 can contain no substitution that has just p conjugates and is commutative with R. Moreover the central of G' can contain only one subgroup of order p, since if t_1 and t_2 corresponded to independent operations of order p of the central of G', some substitution of the form $t_1 t_2^a$ would be commutative with R and would have just p conjugates in G_1. Hence the central of G' must be cyclic.‡

Any substitution of G that corresponds to an invariant operation of G' can give only invariant commutators; and if G' contains an invariant operation of order p^2, any corresponding substitution s of G has not more than p conjugates in G_1, since s^p is invariant in G_1. Moreover s is not invariant in G_1, since

* Loc. cit., vol. 8 (1907), p. 107.
† Ibid., p. 109; and BULLETIN, 2d series, vol. 14 (1908), p. 328.
‡ Burnside, Theory of Groups of Finite Order, pp. 73, 75.

it is not commutative with R^p. Hence in each component of G_1, as the ith one, s_i is contained in an abelian subgroup of index p, and the subgroup of G_1 within which s is invariant is abelian and of index p^2 under G.

We can assume then that the central of G' is of order p, and hence that the central of no succeeding cogredient contains an operation of order greater than p. If G were of class 3, G'' would be of order* p^2 and G' of order p^3. But this is impossible for an irreducible group of degree p^2.† We can assume then that G is of class k, where $k \geqq 4$.

Suppose that t_2 is a substitution of G that corresponds to an invariant operation of G''. Then it must be in G_1,‡ and if $t_{2,i}$ were invariant in the ith component of G_1, $t_{2,j}$ $(j = 1, 2, \ldots, p)$ would be invariant in the jth component, since in $R^{-1}t_2R$ the ith component is the same (except for the names of the variables) as the $(i - 1)$th component of t_2, and since the commutator of R and t_2 is invariant in each component of G_1. Hence either t_2 is invariant in G_1, or $t_{2,i}$ is not invariant in the ith component and gives only invariant commutators. In the latter case the substitutions of the ith component of G_1 that are commutative with $t_{2,i}$ form an abelian group, and hence the substitutions of G_1 that are commutative with t_2 form an abelian group. Moreover this abelian group cannot be of index greater than p^2 under G_1, since t_2 cannot have more than p^2 conjugates under G_1. Hence G contains an abelian subgroup of index not greater than p^3.

We have now to consider the case in which t_2 is invariant in G_1. If under this supposition the central of G'' were of order greater than p, we could assume that every substitution of G that corresponded to an invariant operation of G'' is invariant in G_1. If then t_2 and s_2 were substitutions of G that corresponded to two independent operations of the central of G'', $t_2s_2^{\alpha}$ would, for a suitably chosen value of α, give only invariant commutators in G. But this is impossible. Hence we consider the central of G'' to be of order p.

If now G were of class 4 $(k = 4)$, G''' would be generated by two independent generators each of order p, and hence G_1 would be abelian. But this is impossible in an irreducible group of degree p^2.

* Loc. cit., vol. 3 (1902), p. 351.

† Loc. cit., vol. 7 (1906), p. 67.

‡ Ibid., p. 62.

We now assume that an irreducible group of degree p^2 and order p^m contains an abelian subgroup of index less than p^k, if it is of class k, unless the central of each cogredient up to and including the $(k - j - 1)$th is of order p, and the substitutions of G that correspond to any operation of these centrals are invariant in G_1. In this excluded case, if t_{k-j} is a substitution of G that corresponds to an invariant operation of $G^{(k-j)}$, it is contained in G_1. If it is not invariant in G_1, the substitutions of G_1 that are commutative with it form an abelian subgroup of index not greater than p^{k-j} and G contains an abelian subgroup of index less than p^k. If t_{k-j} is invariant in G_1, it can easily be shown (as in the case $k = 4$) that we need only to consider the case in which the central of $G^{(k-j)}$ is of order p. Hence if our assumption holds for any value of $j (> 1)$, it holds for the next smaller value of j. But we have shown that it holds for $j = k - 2$.

We have assumed throughout that $k > 2$. If $k = 2$, G' is of order p^4 and contains an abelian subgroup of order p^{m-2}.*

We have proved therefore the

THEOREM: *If G is a group of order p^m and class k (>2) that is simply isomorphic with an irreducible group of degree p^2, it contains an abelian subgroup of index less than p^k. If $k = 2$, G contains an abelian subgroup of index p^2.*

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August, 1911.

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* Loc. cit., p. 67, and vol. 3 (1902), p. 343.