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THE PUBLISHED AND UNPUBLISHED WORK OF CHARLES STURM ON ALGEBRAIC AND DIFFERENTIAL EQUATIONS.

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BY PRESIDENT MAXIME BÔCHER.

CHARLES Sturm was born in 1803 at Geneva, then a part of France, and went to Paris at about the age of twenty-one. There he spent the rest of his life and died in 1855, having become a member of the French Academy of Sciences in 1836.

It is not necessary for us to go beyond this bare outline of Sturm's life, since it is not with his worldly fortunes that we shall be concerned. Neither do I propose to give an account of his life-work as a whole.* The brief biography and bibliography prefixed to his posthumous Cours d'Analyse fulfills to some extent both of these purposes. We shall confine ourselves to one branch of investigation pursued by Sturm: the study of the real solutions of algebraic equations and of linear differential equations, both ordinary and partial. It was here that Sturm's most important and suggestive work was done, and it is of interest to try to gain some insight into the relations between the various parts of the subject as they appeared to him.

The papers with which we are concerned may be exhibited in the following table:

* In brief we may say that, besides the investigations with which we shall be concerned, Sturm published

(*a*) An experimental memoir in collaboration with Colladon on the compressibility of liquids.

(*b*) A large number of minor papers, mostly geometrical.

(*c*) Several papers on geometrical optics including a long memoir.

(*d*) Some papers, partly in collaboration with Liouville, on the imaginary roots of equations, which are not without connection with Sturm's work on the real roots of algebraic equations.

THE PAPERS OF 1829, presented to the Academy on the dates given and summarised, as indicated, in the *Bulletin de Férussac:*

May 23. "Sur la résolution des équations numériques." Volume 11, pages 419–422, and volume 12, page 318, footnote.

June 1. No title given. Volume 11, pages 422–424, and volume 12, page 318, footnote. The subject of this memoir is the equation $Ax^\alpha + Bx^\beta + \cdots + Mx^\mu = 0$, where $\alpha, \beta, \ldots, \mu$ are real but not necessarily rational.

June 8. Note. Volume 11, page 425. It is merely stated that this note contains (1) two new proofs of the reality of the roots of the transcendental equations to which the solution of various problems in mathematical physics leads; (2) the general determination of the constant coefficients in the series for representing an arbitrary function between given limits.

July 27. "Sur l'intégration d'un système d'équations différentielles linéaires." Volume 12, pages 314–322.

August 3. "Sur la distribution de la chaleur dans un assemblage de vases." Volume 12, page 322. Nothing but the title of this paper is preserved.

October 19. "Nouvelle théorie relative à une classe de fonctions transcendantes que l'on rencontre dans la résolution des problèmes de la physique mathématique." Volume 12, page 322. Nothing but the title of this paper is preserved unless, as is possible, a brief statement in volume 11, pages 424–425, refers to it.

THE THREE GREAT MEMOIRS.

1835. "Mémoire sur la résolution des équations numériques." *Mémoires des savants étrangers,* volume 6, pages 271–318.

1836. "Mémoire sur les équations différentielles linéaires du second ordre." *Liouville's Journal,* volume 1, pages 106–186. This memoir had been presented to the Academy September 30, 1833, and an abstract of it published in *l'Institut* for November 9, 1833, pages 219–223.

1836. "Mémoire sur une classe d'équations à différences partielles." *Liouville's Journal,* volume 1, pages 373–444. Cf. *l'Institut* for November 30, 1833, pages 247–248.

For the sake of completeness we note that there are also three minor papers of later date.*

It is in the three great memoirs of 1835–36 that Sturm gave its final form to so much of his work as he completed; but the above list strongly suggests, what a closer study amply confirms, that it was in the year 1829 that the great creative period of Sturm's life fell, and that the papers presented to the Academy in that year, so far as they are still accessible, must be examined if we would gain an insight into the lines of thought followed by him in making his great discoveries. In doing this we shall find that certain not uninteresting aspects of his early work find little or no mention in the great memoirs. Most of this early work is preserved to us only in the form of brief abstracts, sometimes even only by ts title, so that some reconstruction becomes necessary. This makes it impossible for us to attain certainty at all points, but perhaps the discussion is not less interesting for this reason.

Sturm's personal and scientific relations to Fourier form an indispensable background to a consideration of the papers presented to the Academy in such rapid succession during the summer of that fruitful year. The two main subjects of Fourier's life work had been the theory of heat and the theory of the solutions of numerical equations. Both of these subjects were carried forward by Sturm, the first in the two memoirs of 1836, the second in that of 1835. But if in the memoirs these tendencies appear quite distinct, we find them, when we turn to the papers of 1829, blended in a most curious and interesting manner.

Fourier's treatise on the solution of numerical equations was not published until 1831, after the author's death; but the manuscript of this work had already in 1829 been communicated to several persons among whom was Sturm, who tells us explicitly in the paper of May 23 what a strong influence it had on his own work.

Fourier had established the theorem that a real algebraic equation of the kth degree, $f(x) = 0$, cannot have more roots in an interval ab, neither of whose extremities is a root, than the difference between the number of variations of sign in the

* Namely: a brief extract of a memoir written by Sturm and Liouville together: "Sur le développement des fonctions en séries . . . ," published in *Liouville's Journal*, vol. 2 (1837), pp. 220–223, and also in the *C. R.*, vol. 4, p. 675; and two papers concerning the real roots of algebraic equations in *Liouville's Journal*, vol. 7 (1842), pp. 132–133, 356–368.

set of functions

$$(1)\qquad f(x),\ f'(x),\ f''(x),\ \ldots,\ f^{[k]}(x),$$

(accents denoting differentiation) at the points a and b. Sturm's theorem, as it is still called, replaces the sequence (1) by

$$(2)\qquad f_0(x),\ f_1(x),\ f_2(x),\ \ldots,$$

which coincides with (1) in the first two places, while each subsequent f_n is the negative of the remainder obtained by dividing f_{n-2} by f_{n-1},

$$(3)\qquad f_{n-2}(x) = q_{n-1}(x)f_{n-1}(x) - f_n(x) \quad (n = 2, 3, \ldots).$$

The advantage of this set over the set (1) is that the difference in the number of variations in (2) at a and b is *precisely equal* to the number of roots between a and b. Since this theorem is given in the first of the notes of 1829 and is elaborated at length in the first of the great memoirs, one might be tempted to suppose that this formed the starting point in Sturm's researches. Fortunately Sturm himself has preserved us from this mistake, for on the closing page of the first memoir of 1836 he tells us that the above theorem was merely a by-product of his extensive investigations on the subject of linear difference equations of the second order. Curiously enough, however, this subject of difference equations is nowhere else alluded to in Sturm's published writings.

The key to this difficulty lies, I feel sure, in the paper of August 3, 1829, of which as has been said only the title is preserved. This memoir is described as more extensive than the one of July 27, which, as one sees from the summary, was not brief. At Sturm's death there was found among his papers a "very extensive" memoir with almost precisely the same title,* which has also never been published. I shall try to show you how this lost paper forms the starting point in Sturm's investigations, and how all his other work which concerns us here grew directly out of it. For this purpose we must first reconstruct at least the general framework of this paper.

* "Sur la communication de la chaleur dans une suite de vases." Cf. Cours d'Analyse, vol. 1, p. xxviii. It is there said: "Ces deux mémoires" (*i. e.*, this one and one on curves of the second order) "sont en état d'être imprimés, et M. Liouville a bien voulu se charger de leur publication." It is to be regretted that this intention was never carried out. Even at this late date the publication would be decidedly interesting if by chance the manuscript could still be found.

Suppose we have a number of vases $P_0, P_1, \ldots, P_n$ placed in any position with reference to one another and filled with various liquids at diverse temperatures. These vases we suppose to be immersed in an atmosphere which circulates freely and thus maintains a constant temperature which we take as the zero of our scale. Each of these vases radiates heat into this atmosphere, and the vases also interchange heat among themselves by radiation. Let us denote the temperature of the vase P_i at the time t by $u\ (i, t)$. The differential equation for the flow of heat is then, if we assume the Newtonian law of radiation,

$$(4) \qquad c(i)\frac{\partial u(i, t)}{\partial t} = k(i, 0)u(0, t) + \cdots + k(i, n)u(n, t) \qquad (i = 0, 1, \ldots, n).$$

Here $c(i)$ is a positive constant depending on the specific heat of the vase P_i; when $i \neq j$, $k(i, j) = k(j, i)$ is a positive constant of proportionality which measures the amount of radiation between P_i and P_j; and finally $k(i, i)$ is written merely as an abbreviation for

$$k(i, i) = -h(i) - k(i, 0) - \cdots - k(i, i-1) - k(i, i+1) - \cdots - k(i, n),$$

where $h(i)$ is a positive constant of proportionality for measuring the radiation of P_i into the atmosphere. It is important for us to understand that the constant $k(i, j)$, when $i \neq j$, depends for its value not merely on the relative positions and the sizes and shapes of the vases P_i and P_j, but that its value is also decreased if one or more of the other vases is so placed as to cut off part of the radiation from P_i to P_j. If these vases are *completely* cut off from each other by intervening vases, the constant $k\ (i, j)$ has the value zero.

I suspect that it was precisely to the problem just indicated and in particular to the system of equations (4) that Sturm first turned his attention. There is, however, evidence (cf. the paper of July 27) that he had also under consideration problems in small vibrations and in celestial mechanics leading to systems of differential equations analogous to (4) but which may be somewhat more general in form. The paper last cited was devoted to the analytical treatment of such systems of linear homogeneous differential equations with constant coefficients. As the chief result is a method of treating the algebraic *characteristic equation* of the system, we postpone any discussion of this paper until later.

It is not, however, in more general, but rather in more special problems that Sturm found his real inspiration. Consider the case in which the vases $P_0, \ldots, P_n$ are arranged in linear sequence* and in such a way that the radiation between two non-consecutive vases is completely cut off by the intervening ones. This case is characterised analytically by all the constants $k(i, j)$ vanishing except those for which the integers i, j are either equal or differ from one another by unity. We may then write equations (4) as follows:

$$
(5)\quad
\begin{aligned}
c(0)\frac{\partial u(0, t)}{\partial t} &= -[h(0) + k(0, 1)]u(0, t) + k(0, 1)u(1, t),\\
c(i)\frac{\partial u(i, t)}{\partial t} &= k(i, i-1)u(i-1, t) - [h(i) + k(i, i-1)\\
&\qquad + k(i, i+1)]u(i, t) + k(i, i+1)u(i+1, t)\\
&\qquad\qquad (i = 1, 2, \ldots, n-1),\\
c(n)\frac{\partial u(n, t)}{\partial t} &= k(n, n-1)u(n-1, t)\\
&\qquad - [h(n) + k(n, n-1)]u(n, t).
\end{aligned}
$$

By the side of the problem in the theory of heat which we have just formulated we may advantageously consider the problem of the small transverse vibrations of a stretched elastic string whose mass is negligible but which is weighted at a number of points by heavy particles. This problem had first been considered one hundred years earlier by John Bernoulli, and for half a century this and equivalent problems had been, in more or less general forms, subjects of investigation by Daniel Bernoulli, Euler, and Lagrange;† but none of these mathematicians had gone beyond the case where the particles

* It is interesting to note that in the title of the manuscript found at the time of Sturm's death, and which may be supposed to be the final form which his memoir took, the phrase "une suite de vases" is used in place of the earlier "un assemblage de vases." This change suggests that as his work developed Sturm desired to give more prominence to this special case; or, indeed, he may have eliminated all consideration of the more general case.

† Cf. Burkhardt's Report in the *Jahresbericht d. deutschen Mathematiker-Vereinigung*, vol. 10 (1901–1908).

are of equal mass and are equally spaced,* and for fifty years the problem had practically remained untouched.

Suppose that the string in its position of equilibrium lies along the axis of x and that particles $P_0, \ldots, P_n$ with masses $c(0), \ldots, c(n)$ lie respectively at the points whose abscissas are $x_0 < x_1 < \cdots < x_n$, and let P_0 and P_n be at the ends of the string. In order to secure the same degree of generality as in the heat problem above mentioned, we assume that during the transverse vibration each particle is drawn back towards its position of equilibrium not merely by the tension of the string but also by an additional force acting towards its position of equilibrium and proportional to the distance from this point. The constant of proportionality here we denote by $h(i)$ in the case of the ith particle, and, calling the tension of the string T, we let

$$k(i, i-1) = \frac{T}{x_i - x_{i-1}}.$$

The distance of P_i from its position of equilibrium at the time t we denote by $u(i, t)$, and we assume that each particle is free to move only in a direction at right angles to the axis of x, and that the whole motion takes place in one plane. Finally we assume that the string always remains so nearly straight that the squares of the sines or tangents of the angles which its pieces make with the axis of x may be neglected.† Then it is readily seen that the equations of motion of the particles become identical with equations (5) *provided we replace the first derivatives in these equations by second derivatives.* It follows that the mechanical problem last mentioned is mathematically almost equivalent to the problem in the theory of heat considered above. That Sturm chose the latter rather than the former, with which he was surely familiar, is, so far as the greater part of his work goes,‡ a matter of slight importance and is prob-

* In all the cases treated during the eighteenth century the particles are either supposed to be acted upon by no external forces, or to be under the influence of gravity acting in the direction of the string. The presence of such an external constant force as this is mathematically equivalent to an unequal spacing of the particles, and will therefore not be explicitly considered by us.

† It should be noticed that we do not assume the ends of the string to be fixed. If we did this, we should have a case strictly analogous to that in which the two extreme vases, P_0 and P_n, in the heat problem are maintained at the temperature zero.

‡ An essential difference occurs only in the work which leads up to the latter part of the second memoir of 1836.

ably to be explained by the fact that the theory of heat was at that moment a more "up to date" subject. The relation between the two problems is however so close that we may be permitted to depart so far from strict historical accuracy as to substitute the vibration problem for the heat problem in our further explanations, since in this way greater concreteness of expression may be gained.

We consider first the simple harmonic vibrations of the string, that is we assume that u has the form

$$u(i, t) = y(i)\,[A \sin \mu t + B \cos \mu t]. \tag{6}$$

By substituting this expression in the equations of motion we find for $y(i)$ the difference equation of the second order

$$\begin{aligned} k(i, i-1)y(i-1) - [h(i) + k(i, i-1) + k(i, i+1) \\ - \mu^2 c(i)]\, y(i) + k(i, i+1)y(i+1) = 0 \end{aligned} \tag{7}$$

or

$$\Delta\{k(i, i-1)\Delta y(i-1)\} + [\mu^2 c(i) - h(i)]y(i) = 0 \tag{7'}$$

together with the terminal conditions

$$k(1, 0)y(1) + [\mu^2 c(0) - h(0) - k(1, 0)]y(0) = 0, \tag{8}$$

$$[\mu^2 c(n) - h(n) - k(n, n-1)]y(n) + k(n, n-1)y(n-1) = 0. \tag{9}$$

The equation (7) has in general no solution other than zero which satisfies both conditions (8) and (9),—it is only for special values of μ that these conditions can both be fulfilled. Consequently we shall disregard at first condition (9), and consider merely the solution of (7) which satisfies (8). This solution obviously contains an arbitrary constant factor, since (7) and (8) are homogeneous. We therefore replace the condition (8) by the two non-homogeneous conditions

$$\begin{aligned} y(0) &= k(0, 1), \\ y(1) &= h(0) + k(0, 1) - \mu^2 c(0), \end{aligned} \tag{8'}$$

which are precisely equivalent to (8) except that (8′) also determines the otherwise arbitrary constant factor, and determines it in such a way that $y(i)$ does not vanish identically.

The solution $y(i)$ determined by (8′) does not in general satisfy

(9). It does, however, for a specified value of μ either satisfy the condition $y(n) = 0$, or a condition of precisely the form (9) where either $c(n)$ or $h(n)$ have in general been replaced by another value.* We may therefore say that for every value of μ the function $u(i, t)$ defined by (6) gives a simple harmonic vibration corresponding either to the mechanical problem we wish to consider or to a modification of this problem which consists either in having the particle P_n held fast, or in a change in the mass of this particle, or in a change in the strength of the force which pulls this particle back to its position of equilibrium. This we shall speak of as the modified problem corresponding to a given value of μ, using this term so that, for the special values of μ above referred to, the original problem itself is the modified problem.

In the plane in which the vibration takes place let us now construct an ordinate of length $y(i)$ at each of the points x_i and connect the extremities of each two successive ordinates by a straight line. The broken line thus formed, which we shall call the line $y(i)$, gives essentially the shape of the string in the simple harmonic vibration we are considering; for, if we multiply by $A \sin \mu t + B \cos \mu t$ all the ordinates of this broken line, and this will evidently not essentially affect its shape, we get a broken line which has precisely the shape of the string at the time t. The points where the line $y(i)$ meets the axis of x thus give the *nodes* of the simple harmonic vibration in question. Either from simple mechanical considerations or from the equation (7) we see that at each node the line $y(i)$ crosses the axis. Consequently, since each of the quantities $y(0)$, $y(1)$, $\cdots$, $y(n)$ is obviously a continuous function of μ (in fact a polynomial in μ^2), the nodes also vary continuously with μ, never suddenly appearing or disappearing except at the extremity x_n of the string.

There can be very little doubt that at this point Sturm, by a simple manipulation of equation (7′) which we will not stop to give here†, established the important fact, which may easily

* This is true even in the case $\mu = 0$ provided we replace $h(n)$ by a negative quantity. In all other cases positive quantities may be used in place of $c(n)$ and $h(n)$.

† Cf. Porter, *Annals of Mathematics*, second series, vol. 3 (1902), p. 55. In the article here cited Professor Porter, at my suggestion, reconstructed a part of Sturm's researches on difference equations without, however, considering either the vibration problem or the heat problem. Cf. also the article by Professor Porter's pupil, Miss Merrill, *Trans. Amer. Math. Soc.*, vol. 4 (1903), p. 423.

have been suggested to him by the mechanical problem itself, that as μ^2 increases the abscissa of each node decreases, new nodes appearing one by one at the point x_n.

It is here that Sturm must have noticed the connection with Fourier's theorem concerning the roots of algebraic equations. To establish this connection we need merely to observe that the number of nodes for a given value of μ^2 (we shall write for convenience $\lambda = -\mu^2$) is simply the number of variations of sign in the set of polynomials in λ

$$y(n), y(n-1), \cdots, y(0). \tag{10}$$

Consequently what we have said above is equivalent to saying that the number of roots of the polynomial $y(n)$ between two negative values of λ is precisely equal to the difference between the number of variations in the system (10) for these two values of λ. Sturm found, therefore, that for the particular polynomial $y(n)$ he was in possession of a sequence of polynomials of descending degrees (since $y(i)$ is a polynomial of the ith degree in λ), which served *perfectly* the purpose which the sequence of derivatives serves imperfectly in Fourier's theorem. He must then have asked himself to what properties of the polynomials (10) this fact is due, and have seen that just three properties were used:

(*a*) The last polynomial, $y(0)$, is a constant not zero.

(*b*) When one polynomial vanishes the two adjacent ones have opposite signs. This was an immediate consequence of (7), but would follow in the same way if instead of (7) the y's satisfied any difference equation

$$L(i)y(i+1) + M(i)y(i) + N(i)y(i-1) = 0, \tag{11}$$

where L and N always have the same sign.

(*c*) The nodes increase with λ. Far less than this, however, would be sufficient for our present purpose, namely that when a node lies at x_n it decrease as λ decreases. This would in particular be the case if $y(n)$ had no multiple root and $y(n-1)$ were simply the derivative of $y(n)$, for this is precisely the property of the derivative on which Fourier's work is based.

We may then suppose that Sturm said to himself: Starting with any polynomial without multiple roots, $f(\lambda)$, in place of $y(n)$, and taking in place of $y(n-1)$ the derivative $f'(\lambda)$, how can I form a sequence of polynomials $f_2(\lambda), f_3(\lambda), \ldots,$ to take the place of $y(n-2), y(n-3), \ldots,$ which satisfy a relation of the form (11) and of which the last is a constant not zero? This question once formulated, the method of successive divisions and reversal of sign of the remainder, leading to equation (3) which is merely a special case of (11), would readily suggest itself, and Sturm's theorem in its most familiar form was found.

If our surmises so far are correct, it follows that, even at this early date Sturm must have been well aware that any sequence of functions having a small number of easily specified properties would serve the purpose of his theorem just as well as the sequence (2); so that to call such more general sequences Sturmian sequences, as is now done,* is even from a strictly historical point of view entirely suitable. Our belief that Sturm was familiar with this more general point of view need not, however, rest entirely on the line of reasoning so far explained. Not only does he show in his memoir of 1835 how other Sturmian sequences besides (2) may be formed;† but more particularly his paper of July 27, 1829, to which reference has already been made, is mainly devoted to the formation for a special algebraic equation of a Sturmian sequence which is very different from the sequence (2). If we use the notation of determinants, which Sturm does not use, the equation in question is

$$(12)\qquad \begin{vmatrix} g_{11}\lambda + k_{11} & g_{12}\lambda + k_{12} & \cdots & g_{1n}\lambda + k_{1n} \\ g_{21}\lambda + k_{21} & g_{22}\lambda + k_{22} & \cdots & g_{2n}\lambda + k_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ g_{n1}\lambda + k_{n1} & g_{n2}\lambda + k_{n2} & \cdots & g_{nn}\lambda + k_{nn} \end{vmatrix} = 0,$$

where $g_{ij} = g_{ji}$, $k_{ij} = k_{ji}$, and where the g's and k's are real and the former are the coefficients of a non-singular definite quadratic form. Sturm falls here into the error into which Laplace and Lagrange had fallen before him, and which was first corrected by Weierstrass in 1858, of thinking that this equation can have no multiple roots. He gives, however, the correct theorem that the roots are all real; and, what is important for us here, he states that the determinant in (12) and the polynomials obtained by striking off from it the last 1, 2, 3, $\cdots$ rows and columns form

* Cf. Weber's Algebra, 2 ed., vol. 1, p. 303.

† Cf. also the closing lines of his abstract of May 23, 1829.

a Sturmian sequence,* provided each of these polynomials is multiplied by such a power of -1 as to make the coefficient of its leading term positive.†

The paper of May 23, 1829, subsequently published as the memoir of 1835, did more than all of his other papers together to win for its author recognition both in France and throughout Europe. It appears to us here in its true light,‡ as a digression from his investigations in the domain of mechanics and mathematical physics. This digression was indeed carried a little farther, as the paper of June 1 testifies. Here it is shown how Fourier's methods can be applied with very slight change to obtain an upper limit for the number of real roots in a given interval for the type of transcendental equation there considered. It is clear, moreover, from a brief remark near the middle of page 424 that Sturm was here also in possession of a method of forming a Sturmian sequence.

Let us now return to the problem of the vibrating string which we were considering above. We saw that as μ^2 increases the abscissa of each node decreases. Now from (8′) and (7′) it may readily be inferred that when $\mu = 0$,

$$0 < y(1) < y(2) < \cdots < y(n),$$

so that here there is no node. On the other hand from (8′) and (7) we see that $y(i)$ is a polynomial of the ith degree in μ^2 whose leading coefficient has the sign of $(-1)^i$. Consequently, for very large values of μ^2 we have a node in each of the intervals

$$x_i < x < x_{i+1}, \qquad (i = 0, 1, \cdots, n),$$

that is, we have the maximum possible number of nodes, namely n. Accordingly as μ^2, starting with the value zero, increases, we have at first no node, then for a while one node,

* This is true only with the qualification that no two of these polynomials have a common root; cf. Weber's Algebra, 2 ed., vol. 1, p. 308. This necessary qualification is not mentioned in Sturm's abstract, though it is by no means impossible that it may have been given in the extended memoir.

† In other words, these factors are all $+1$ or alternately $+1$ and -1 according as the quadratic form of which the g's are the coefficients is positive or negative.

‡ There are further details elaborated in this memoir to which it is not necessary for us to refer.

then for a certain interval two nodes, etc., until finally the nth node appears at x_n and from that point on we constantly have n nodes. We thus see that there are just n positive values of μ^2, for which a node lies at x_n, that is, that the equation in μ^2, $y(n) = 0$, has n distinct positive roots, and consequently, since it is of the nth degree, that it has no imaginary or negative or multiple roots. If we denote the roots arranged in order of increasing magnitude by $\mu'^2, \mu''^2, \cdots, \mu^{[n]2}$ it is clear from what has been said that for positive values of μ^2 $y(i)$ has just k nodes ($k \leqq n$) in the interval $x_0 < x < x_n$ when and only when $\mu^{[k]2} < \mu^2 < \mu^{[k+1]2}$, where, for convenience, we let $\mu^{[0]} = 0$, $\mu^{[n+1]} = +\infty$.

The next step is to show that in this interval there exists one and only one value of μ^2 for which $y(i)$ satisfies the condition (9), and this follows readily from the fact* that, when $y(n) \neq 0$, $y(n-1)/y(n)$ increases as μ^2 increases, and hence increases from $-\infty$ to $+\infty$ as μ^2 increases through the interval we are now considering. This establishes the following:

THEOREM OF OSCILLATION.† There exist just $n + 1$ values of μ^2, all real and positive, for which the difference equation (7) has a solution, not identically zero, which satisfies the terminal conditions (8), (9). Denoting these values, arranged in order of increasing magnitude, by $\mu_0^2, \mu_1^2, \cdots, \mu_n^2$, the solution $y_k(i)$ of (7) corresponding to μ_k^2 and satisfying (8), (9) has exactly k nodes.

We obtain in this way $n + 1$ simple harmonic vibrations of the weighted string, which differ from one another in the number of their nodes. The most general motion of the string will be obtained by compounding these simple harmonic vibrations. Here the formulæ, which are readily obtained, for making the vibration correspond to arbitrarily given initial conditions are closely analogous to the well known ones for the representation of periodic functions at n equally spaced points by a finite trigonometric series. These latter formulæ had been obtained by Lagrange in considering a special case of the vibration problem with which we have been concerned.

In precisely the same way, the problem of the distribution of

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* Cf. Porter, loc. cit. This fact is also necessary in the proof, referred to above, that the nodes decrease as μ^2 increases.

† See the articles by Porter and Miss Merrill above cited. This theorem, never published by Sturm, is the prototype of the important and still increasing class of theorems to which Klein attached this name more than fifty years later.

heat in a row of vases is solved by the formula

$$(13) \qquad A_0 e^{-\lambda_0 t} y_0(i) + A_1 e^{-\lambda_1 t} y_1(i) + \cdots + A_n e^{-\lambda_n t} y_n(i),$$

where $\lambda_k = -\mu_k^2$. Here also the coefficients A_k can be determined so as to correspond with given initial conditions. It was, however, not merely the analytic solution of this problem which interested Sturm, but even more, perhaps, a discussion of the properties of the solution.

If we mark the vases at any moment $+$ or $-$ according as their temperatures are above or below the temperature of the surrounding atmosphere, it is obvious from physical considerations that as time passes the number of variations of sign in the sequence cannot increase. This fact may be established analytically by means of the equations (5) of which the function (13) is a solution, and at the same time we can see precisely how the number of variations decreases. For a fixed value of t we plot the function (13) as a broken line, precisely as above we plotted the line $y(i)$. This line, however, unlike the line $y(i)$, does not necessarily cross the axis of x at every point where it meets it; indeed it may meet the axis of x not merely at isolated points but it may also coincide with it throughout a whole segment extending between two points x_i. Let us call each of the isolated points and segments where the broken line representing (13) meets the axis of x a node of (13). To each node we attribute a multiplicity as follows: If the node lies at, or reaches up to, one of the end-points x_0 or x_n, we take as its multiplicity the number of points x_i contained in it. Otherwise we take either this number or a number one greater, in such a way that the multiplicity shall be odd or even according as in passing through the node the function (13) does or does not change sign. This convention is justified by the fact, readily established by means of equations (5), that such multiple nodes can occur only for isolated values of t; and that for values of t a little smaller than such a value, the function (13) has exactly k simple nodes in the neighborhood of the point or segment where a k-fold node is to appear.* Now the fundamental fact here, which also follows from (5), is that as t increases through a value for which there is a node of multiplicity k, the simple nodes, after coalescing to form the multiple node, all disappear,

* We exclude here and in what follows the possibility that the function (13) vanish identically; or, what is the same thing, we assume that not all the A's are zero.

leaving no node at all in the neighborhood of the point or segment in question, except in the one case of a node of odd multiplicity which does not lie at or reach up to one of the end-points x_0 or x_n; in which case just one simple node remains in the neighborhood in question.

From these considerations it may readily be inferred that the number of nodes of the expression

$$A_p y_p(i) + A_{p+1} y_{p+1}(i) + \cdots + A_q y_q(i), \tag{14}$$

where $A_p \neq 0$, $A_q \neq 0$, $0 \leqq p < q \leqq n$, cannot be less than p or greater than q, a multiple node being counted at pleasure either once or as often as its multiplicity indicates. For if, by introducing exponentials, we modify (14) into an expression of the form (13), the nodes of (14) appear merely as the nodes of (13) when $t = 0$, and the number of such nodes lies, by what was said above, between the number of nodes of the expression (13) in question for a very large negative, and the number for a very large positive value of t. For such extreme values, (13) coincides very nearly with constant multiples of $y_p(i)$ and $y_q(i)$ respectively.

While it does not seem likely from the scanty evidence which Sturm has left us that these latter considerations were all familiar to him in 1829, they can hardly fail to have been in his possession four years later, and it is not unlikely that if the manuscript which was among his papers at the time of his death could be recovered, it would be found to contain a systematic exposition of them along with the other matters we have touched upon.*

Two lines for further investigation now naturally presented themselves. One of these consisted in replacing the difference equation (7) and the boundary conditions (8), (9) by the more general relations of the same form

$$\begin{aligned} \Delta\{K(i)\Delta y(i)\} - G(i)y(i+1) &= 0 \quad (K(i) > 0), \\ K(0)\Delta y(0) - hy(0) &= 0 \text{ (or } y(0) = 0), \\ K(n-1)\Delta y(n-1) + Hy(n-1) &= 0 \text{ (or } y(n) = 0). \end{aligned} \tag{15}$$

* We note in passing that if we equate the expression (13) to zero and assign to i a particular value, the equation thus obtained is precisely of the form considered in the paper of June 1, 1829 concerning which we have already spoken. Cf. the introductory remarks in the summary of that paper.

If here we assume that the quantities $K(i)$, $G(i)$, h, H are continuous increasing functions of λ, the results relating to equation (7) and also their proofs admit of ready extension to the system (15).

On the other hand we may pass over from *difference* to *differential* equations by allowing the integer n to become infinite, the points x_0 and x_n however remaining fixed. In this way the various functions of the integral argument i become functions of a continuous argument x. We thus pass from the massless string weighted by n distinct particles to the string whose mass is continuously but unequally distributed throughout its extent, and from the radiation of heat in a row of vases to the conduction of heat in a heterogeneous bar. It is this latter problem which forms the subject of Sturm's second great memoir of 1836, while the extension of the results concerning the difference equation (15) to the differential equation

$$\frac{d}{dx}\left(K(x)\frac{dy}{dx}\right) - G(x)y = 0$$

is the subject of the first memoir of that year. It is worthy of notice that in both cases Sturm used the method of passing by a limiting process from a difference to a differential equation merely as a heuristic one, making indeed hardly a mention of it in the final memoirs, and treating the differential equations *directly* by methods which are the immediate generalizations of those he had used for the difference equations in his unpublished work.* A careful examination of the abstracts published in 1829 in the *Journal de Férussac* will show that all this, at least so far as it relates to the heat problem, was in his mind even at that early date.† It was probably developed to some extent in the unpublished paper of October 19.

It is not my purpose to discuss here the two great memoirs of 1836, although the richness of their detail tempts one to linger over many points which have, it is to be feared, rather escaped the notice of mathematicians. This richness of detail probably reflects a similar quality in the earlier unpublished

* Cf. Fredholm's derivation of the theory of integral equations as a limiting case from the theory of a system of linear algebraic equations. Like Sturm, Fredholm used this limiting process merely as a heuristic one for deriving both the results and the methods for their proofs.

† There is no evidence to show whether the more general difference equation (15) was considered at this time or only at a slightly later date.

researches, and much of the detail there could be readily reproduced. In these days, when new methods are being suggested for obtaining a few of the fundamental results of Sturm in their simplest forms, it is not out of place to remark that if one were to cut away from Sturm's memoirs everything except what is necessary to obtain *these* results, the few pages that would be left would in brevity, rigor, and directness easily stand comparison with anything which has so far been suggested to replace them.

Coupled with Sturm's name in all of this work on differential equations one often finds the name of his young friend Liouville. It is true that Liouville's work on these matters was hardly inferior in originality and power to that of Sturm himself; but it must be remembered that Sturm's work was practically complete, even to the writing of the two great memoirs, before Liouville's began, and that, except for alternative proofs which the latter gave for some of Sturm's results, and for a genial extension to certain differential equations of higher order, his work dealt with a single problem, of fundamental importance it is true, which had not been treated by Sturm,* namely the proof that the development of an arbitrary function which occurs in Sturm's papers is valid. We may therefore fairly speak of the Sturm-Liouville development according to normal functions, but these normal functions themselves, and almost everything relating to their theory,† are due to Sturm alone.

I have tried to show you how all of Sturm's most important work flowed naturally from his treatment of a single physical problem, not very important in itself perhaps, certainly of no great generality or largeness of scope. Sturm's genius showed itself first in his method of handling the problem where such purely formal skill as one associates with the names of Lagrange or Poisson is less in evidence than a constant intuitional visualization of the problem combined with a sense of accuracy uncommon in his day; secondly in his perception of the relation of this problem to other questions, and to the way in which he followed up his work into adjacent fields. The power of generalizing is not rare, as the huge bulk of our current mathe-

* One paper on this subject written in conjunction with Liouville is preserved to us in abstract (*Liouville's Journal*, vol. 2, p. 220) but was written after Liouville's first work on this subject.

† An exception should be made here of the asymptotic expressions for these functions for large values of the index. These important expressions are due to Liouville.

matical literature sadly reminds us; but one who like Sturm can seize on the important and simple modifications of a given problem has certainly one of the most essential elements of mathematical greatness.

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A SENSUOUS REPRESENTATION OF PATHS THAT LEAD FROM THE INSIDE TO THE OUTSIDE OF AN ORDINARY SPHERE IN POINT SPACE OF FOUR DIMENSIONS WITHOUT PENETRATING THE SURFACE OF THE SPHERE.

BY PROFESSOR C. J. KEYSER.

(Read before the American Mathematical Society, April 28, 1911.)

THE logical or analytic existence of such paths—their existence in and for thought as distinguished from intuition or imagination—has been long familiar to every one, and may be made evident even to a freshman, so simple is the sufficient algebraic argument. But all efforts to envisage the paths are defeated completely.

It is the purpose of this note to show how the existence of the paths may be made evident to the intuition and even to the senses of sight and touch. The purpose is achieved by a simple transformation correlating the points of 4-space S_4 with the spheres of ordinary space S_3, including all spheres of real center and pure imaginary radius. In this way unintuitable situations in S_4, like that presented by the paths in question, are represented by intuitable analytic equivalents in S_3, and these equivalents may be rendered sensible by easily constructible physical models.

The simplest possible correlation of the kind in question is that in which the point (x, y, z, w) of S_4 and the sphere (of S_3) having (x, y, z) for center and $\sqrt{w}$ for radius shall be a pair of correspondents.

The representative in S_3 of a lineoid (an ordinary 3-space) $Ax + By + Cz + Dw + E = 0$ of S_4 is a linear complex of spheres such that, if (x_1, y_1, z_1, w_1) be a point of the lineoid, the