It appears that thirteen institutions of the twenty-four adhere strictly to this rule, while in at least five more the requirement is optional with the instructor in charge or the head of the department. In three or four others a thesis is never presented for the master's degree.

The last topic considered is that of examinations. The report mentioned recommends either a final test covering all the graduate courses taken by the student, or the requirement that a high grade be obtained in term examinations in each course. The former alternative is the more usual one, —in fact it obtains at eighteen of the twenty-four institutions considered, while in several of the others either procedure is possible. At Harvard one of the two highest of the four passing grades in each course is the sole requirement. The final examination may be either oral or written, — practice seems to be fairly evenly divided in this matter, — and if oral, it is usually conducted by a committee of instructors from the departments involved.

There are smaller and less reputable institutions in which the requirements are lower, and in fact some where all requirements are merely nominal. The present report has aimed only to present the state of affairs in the institutions which are really prepared to confer the degree.

GEOMETRIC TRANSFORMATIONS.

Geometrische Transformationen. Zweiter Teil: die quadratischen und höheren birationalen Punkttransformationen. By KARL DOEHLEMANN, associate professor of mathematics at the University of Munich. Leipzig, Göschen (Sammlung Schubert, XXVIII), 1908. viii + 328 pages and 59 figures.*

AT the time this volume was written there was no book devoted to the subject of its title. The next year appeared the fourth volume of Professor Sturm's treatise † and several parts of the Encyklopädie, so that now we are fairly well supplied with an introduction to this most fundamental study. And the two books do not overlap; the larger work is synthetic and replete with geometric illustrations, while that under review is

* The first part of Professor Doehlemann's treatise appeared in 1902; it was reviewed in the BULLETIN by Professor Gale (vol. 10, pp. 512–515).

† See the review in the BULLETIN, volume 17 (1910), pp. 150–154.

analytic, elementary, and well adapted to its purpose of serving as a finder into the literature and problems of birational transformations. Perhaps it would have been well to have the name Cremona transformations in the title, for no mention is made of the geometry on a given curve or surface, hence such concepts as point groups, series, residuation, adjoint curves, etc., are entirely absent, as is also the entire theory of multiple correspondences and higher involutions. The selection of a restricted field is a wise one; to treat adequately the subject of birational transformations in this broader sense requires several volumes, and goes beyond the task imposed in the Schubert collection. In fact it is the introduction that is particularly needed. If a reader has a thorough knowledge of Cremona transformations, he is well prepared to study the geometry on a curve or surface by the Brill-Noether method.

The book is divided into two nearly equal parts, the first treating of the plane, the second of space. The first chapter introduces quadratic transformations by means of two pairs of projective pencils of lines, first synthetically, then algebraically. The transformation of a given curve is discussed for a large number of special cases, including the images of an extensive list of singular points. This part is developed synthetically only, thus not furnishing the reader with complete weapons for the resolution of the singularities of a curve whose equation is given. The general bilinear equations are now shown to define the same transformations; the cases in which two or three of the vertices of the fundamental triangle approach coincidence being briefly considered. Finally, the case where the fundamental triangles of the two planes coincide and the condition for involution are discussed. At the end of this and of each subsequent chapter is a list of the memoirs bearing on the topics considered. These lists are not complete, and a few conspicuous absences are noticed, but on the whole they are well selected and give a very good idea of the literature and of the unsolved problems.

Now follow two chapters (pages 50–134) on metrical properties based upon the case in which two of the vertices of the fundamental triangle are the circular points at infinity. Foci, minimum lines, reciprocal radii, orthogonal circles, and analagmatic curves are discussed in a very elementary manner, most concepts being developed independently, then shown to belong under the preceding projective treatment. A number of applications to mechanics, inverting straight lines into circles, are considered, and a number of properties of bicircular quartics and of circular cubics are given. The second of these chapters considers the bilinear relation between two complex variables, the development being similar to that given in most books on the theory of functions. It includes conformal mapping and gives a number of elementary examples. While it is interesting to note the position of this transformation as a particular case of those mentioned before, it seems hardly justifiable to devote over one tenth of a book concerned with geometric transformations to the discussion of a case whose main use is in analytic problems, and which has been treated repeatedly elsewhere. This, however, is the only case in which the perspective of relative importance (for geometric purposes) does not seem to be preserved in the best way. Even here it might be said that all the steps of the general case can be understood most easily by a careful study of the simple one.

Chapter IV, on the general birational transformation of the plane, commences from an entirely different point of view. If

$$x_i' = \phi_i(x_1, x_2, x_3)$$

defines a rational function of the x_i, under what conditions can the equations be solved rationally for x_i in terms of the x_i'? In the search for the answer to this problem we naturally find the Cremona net, meet with fundamental curves and fundamental points, and discover a number of relations existing among them. The Jacobian of the Cremona net is shown to consist of all the fundamental curves of the transformation. A table of all the forms up to $n = 10$ is given, n being the order of the rational curve, image of a straight line. The theorem that every Cremona transformation can be resolved into quadratic inversions is proved in detail when the basis points are distinct; mention is made of the modifications necessary when two or more of these approach coincidence, and the memoirs treating of these cases are cited.

The first part closes with a rapid glance over the newer problems, most of the theorems being given without proof. Since the name of the author appears frequently in the literature of these sections, he is particularly well fitted to prepare such a survey. The involutorial transformations are discussed in some detail, in particular the Jonquières transformations. The results of the researches of Bertini regarding the number of types of irreducible involutions are given, and a method of obtaining all except that of order 17 is added. A section on periodic transformations gives a résumé of the results of Kantor, discusses conjugate and commutative transformations but does not give any geometric criteria. The concept of finite groups of Cremona transformations makes still further demands on the algebra of substitution groups; the results obtained by Wiman are given without proof. A short statement concerning continuous groups is also made, principally to refer to the literature. Whether there are curves which remain invariant under an infinite discontinuous group is not mentioned.

In the second part, the quadratic transformations of space are introduced by means of two reciprocal systems. The treatment is synthetic, and includes a discussion of fundamental elements and of the image of a general curve and a general surface. Then follows an algebraic development of the same idea, the two together making a symmetric and well balanced treatment. Particular cases are discussed briefly but not exhaustively. One misses the inverse of the general quadratic transformation. It is introduced later, but here the reader meets the sole statement that the inverse of a transformation may be of different order than that of the direct operation. The generalized stereographic projection and the statement that the image of a plane is a surface which can be rationally mapped upon a plane fittingly close the chapter.

Now follow two chapters (pages 201–286) on the particular case in which the fundamental conic is the absolute circle at infinity. As in the plane, the discussion is very elementary and proceeds algebraically with non-homogeneous coordinates. Besides the geometry of reciprocal radii a number of applications are made, particularly to the simple stereographic projection and Mercator's chart. Then follows a systematic geometry of the sphere, including complexes, congruences, depiction into hyperspace, and the generation of Dupin's cyclides. Thus far everything is linear; the discussion of the quadratic equation is much more brief. No mention is made of the higher geometry of the sphere, which contains perhaps the most elegant example of birational transformation in the whole field. The excuse may be made that in the elementary orthogonal case we are still dealing with a point transformation, whereas in the general case we are not. On the other hand the entire treatment is in terms of the sphere as element of space, and its coordinates are interpreted as point coordinates in hyperspace.

No apology need be made for this long digression into the geometry of the sphere. Here the purpose is a geometric one and the discussion keeps well within the limits set by the book. The surfaces which naturally appear in this way can be interpreted projectively. If, finally, the line-sphere transformation be applied, we have the Kummer surface and as particular cases the Steiner surface and the quartic ruled surfaces. Together they comprise practically all the quartic surfaces that we know anything about. A short section on electric images contains a list of most of the important memoirs devoted to it.

Chapter VIII, on cubic and higher birational transformations of space, begins with three bilinear equations analogous to the pair previously treated in the plane. It is exceptionally well written and contains all the important principles for the discussion of the (3, 3) transformation, that is, that in which planes go into cubic surfaces, and the inverse is of the same form. By means of these principles it is easy and natural to discuss the mapping of a general cubic surface on the plane. In the treatise of Sturm this depiction precedes the consideration of the cubic transformation, but the treatment of our author seems more natural and is much easier to follow. The details are not given, but three different depictions are outlined.

The last section contains a brief treatment of the general birational transformation, including a discussion of the fundamental points, curves, and surfaces. The monoidal case being given as an example, a sketch is added regarding the method of procedure in the determination of new transformations, and in the corresponding depiction of a surface upon a plane.

Of the few typographical errors noted the only ones that may cause confusion are found on pages 140 and 141. The letter i is there used as symbol of summation in two different senses, the first running over the three coordinates, and the other over the h fundamental points of the plane. In the resulting formulas these two symbols are curiously combined.

In the book under review Professor Doehlemann has admirably fulfilled the purpose aimed at in the Sammlung Schubert.

VIRGIL SNYDER.