

THE APRIL MEETING OF THE AMERICAN
MATHEMATICAL SOCIETY.

THE one hundred and thirty-third meeting of the Society was held in New York City on Saturday, April 27, 1907. The April meeting is usually well attended, and ranks next in importance to the annual and the summer meetings. The present occasion was especially marked by Professor W. F. Osgood's presidential address, the subject of which was "The calculus in our colleges and technical schools." Under this attracting influence the attendance exceeded all previous records for the April meetings, reaching about seventy persons including the following sixty-one members of the Society :

Miss Grace Andrews, Professor G. A. Bliss, Professor Joseph Bowden, Professor E. W. Brown, Dr. W. H. Bussey, Dr. J. E. Clarke, Miss Emily Coddington, Professor F. N. Cole, Miss E. B. Cowley, Professor T. W. Edmondson, Professor L. P. Eisenhart, Professor F. C. Ferry, Professor T. S. Fiske, Dr. F. L. Griffin, Mr. G. W. Hartwell, Dr. G. W. Hill, Dr. A. M. Hildebeitel, Dr. A. A. Himowich, Professor E. V. Huntington, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. G. H. Ling, Mr. Joseph Lipke, Mr. L. L. Locke, Dr. W. R. Longley, Professor E. O. Lovett, Mr. E. B. Lytle, Professor T. E. McKinney, Professor Max Mason, Dr. R. L. Moore, Professor Richard Morris, Professor W. F. Osgood, Dr. F. M. Pedersen, Miss Amy Rayson, Mr. H. W. Reddick, Dr. R. G. D. Richardson, Miss S. F. Richardson, Miss I. M. Schottenfels, Mr. L. P. Sicheloff, Professor C. S. Slichter, Dr. Clara E. Smith, Professor D. E. Smith, Mr. F. H. Smith, Professor P. F. Smith, Dr. R. P. Stephens, Dr. C. E. Stromquist, Dr. W. M. Strong, Professor H. D. Thompson, Mr. M. O. Tripp, Professor H. W. Tyler, Dr. A. L. Underhill, Mr. C. B. Upton, Professor J. M. Van Vleck, Professor Oswald Veblen, Mr. H. E. Webb, Professor H. S. White, Miss E. C. Williams, Professor E. B. Wilson, Professor J. E. Wright, Professor J. W. Young.

The President of the Society, Professor H. S. White, occupied the chair. The Council announced the election of the following persons to membership in the Society: Dr. Alfred Ackermann-Teubner, Leipzig, Germany; Dr. J. W. Bradshaw, University of Michigan; Professor H. E. Cobb, Lewis

Institute, Chicago, Ill.; Mr. S. A. Corey, Hiteman, Ia.; Professor Floyd Field, Georgia School of Technology; Mr. G. W. Hartwell, Columbia University; Chancellor C. C. Jones, University of New Brunswick; Mr. Joseph Lipke, Columbia University; Professor Francis Regis, Christian Brothers College, St. Louis, Mo.; Mr. H. P. Stellwagen, Yeatman High School, St. Louis, Mo. Seven applications for admission to the Society were received. The total membership on May 1 was 560.

Following the plan announced in the Secretary's report of the February meeting, abstracts of the papers, as far as available, had been printed and issued with the programme in advance of the meeting. In this way it is hoped to secure a more intelligent interest in the papers and to promote criticism and discussion.

The date of the summer meeting, to be held at Cornell University, was fixed for Thursday-Friday, September 5-6.

The following papers were read at the April meeting:

- (1) Professor G. A. BLISS: "A new form of the simplest problem of the calculus of variations."
- (2) Professor R. D. CARMICHAEL: "Multiply perfect even numbers of five different primes" (preliminary communication).
- (3) Professor L. P. EISENHART: "Transformations of surfaces whose lines of curvature are represented on the sphere by an isothermal system."
- (4) Dr. F. L. GRIFFIN: "The variation of the apsidal angle in certain families of central orbits."
- (5) Dr. F. L. GRIFFIN: "The solutions of central force problems as functions of the constant of areas."
- (6) Dr. F. L. GRIFFIN: "Note on a simple example of a central orbit with more than two apsidal distances."
- (7) Professor G. A. MILLER: "Note on the commutator of two operators."
- (8) Professor J. E. WRIGHT: "Arrangement of ovals of a plane sextic curve."
- (9) Professor W. F. OSGOOD, Presidential address: "The calculus in our colleges and technical schools."
- (10) Miss I. M. SCHOTTENFELS: "Group matrices."
- (11) Dr. C. E. STROMQUIST: "An inverse problem of the calculus of variations."
- (12) Dr. R. G. D. RICHARDSON: "On the integration of a series term by term."
- (13) Dr. A. L. UNDERHILL: "Invariants of the function

$F(x, y, x', y')$ under point and parameter transformations connected with the calculus of variations."

(14) Professor EDWARD KASNER: "The motion of particles under conservative forces."

(15) Professor EDWARD KASNER: "Isogonal and dynamical trajectories."

(16) Professor P. L. SAUREL: "On the distance from a point to a surface."

(17) Professor T. E. MCKINNEY: "On concyclic quantities."

(18) Professor T. E. MCKINNEY: "On continued fractions representing quadratic irrationalities."

(19) Professor G. A. MILLER: "Groups generated by n operators each of which is the product of the $n - 1$ remaining ones."

In the absence of the authors the papers of Professor Carmichael, Professor Miller, Professor Saurel, and Professor McKinney were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. The simplest problem of the calculus of variations in the plane has been studied in two different forms. In the earlier one the curves considered are represented by equations $y = y(x)$, where $y(x)$ is a single-valued function, and the integral to be minimized is

$$(1) \quad I = \int f(x, y, y') dx.$$

In the later form, which seems first to have been studied in detail by Weierstrass, the curves are taken in parametric representation $x = \phi(t)$, $y = \psi(t)$, and the integral in the form

$$(2) \quad I = \int F(x, y, x', y') dt,$$

where the function F is positively homogeneous in the derivatives x' and y' . Both methods are open to objections, the former because it is inconvenient to apply to curves for which the function $y(x)$ is not single-valued or for which the slope becomes infinite, and the latter on account of the homogeneity condition $F(x, y, \kappa x', \kappa y') = \kappa F(x, y, x', y')$, ($\kappa > 0$), which with its consequences must be kept in mind and frequently applied.

In a paper entitled "A generalization of the notion of angle" (*Transactions of the American Mathematical Society*, volume 7

(1906), page 184), Professor Bliss has introduced a third form of the problem in which the integral involves only invariants of the curve under change of parameter representation. The integral there given is

$$(3) \quad I = \int f(x, y, \tau) ds,$$

where s is the length of arc along the curve, and τ is the angle which the tangent to the curve makes with the x -axis, defined by the equations

$$\cos \tau = \frac{x'}{\sqrt{x'^2 + y'^2}}, \quad \sin \tau = \frac{y'}{\sqrt{x'^2 + y'^2}}.$$

The integral (2) may be easily put into the form (3), for on account of the homogeneity of F ,

$$F(x, y, x', y') = F(x, y, \cos \tau, \sin \tau) \sqrt{x'^2 + y'^2}.$$

But in (3) the function $f(x, y, \tau)$ need not be periodic as is the case in the right member of the last equation. Besides this generalization, which is important in applications to geometry such as Hamel's (see *Mathematische Annalen*, volume 57 (1903), page 231), the use of the integral in the form (3) avoids both of the objections made above to the other methods.

In the author's paper referred to above only the Euler equations and the equations defining transversality, which were necessary in finding the generalization of angle, were developed. In the present paper it is proposed to complete the theory of the problem in the form (3) by deriving the first and second variations and the functions which are commonly used in proving the usual necessary conditions in the calculus of variations (see Bolza, *Lectures on the calculus of variations*, pages 123, 133, 138; also Osgood, *Annals of Mathematics*, series 2, volume 2 (1901), pages 105 ff). The derivation of the conditions themselves can be made as is usual in the books on the subject.

2. By a rather difficult process, which he hopes to be able to simplify later, Professor Carmichael continues his investigations into the existence of multiply perfect numbers. In the present paper he takes up the case of seven numbers having only five different prime factors, finding that there are but four such numbers which are multiply perfect. One of these, $2^{13} \cdot 3 \cdot$

$11 \cdot 43 \cdot 127$, is of multiplicity 3; the others, $2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$, $2^7 \cdot 3^3 \cdot 5^2 \cdot 17 \cdot 31$, $2^9 \cdot 3^3 \cdot 5 \cdot 11 \cdot 31$, are of multiplicity 4.

3. Ten years ago Thybaut* showed that minimal surfaces can be transformed into minimal surfaces in a manner analogous to the Bäcklund transformations of pseudospherical surfaces. A minimal surface and its Thybaut transform are the focal sheets of a W -congruence. The two minimal surfaces adjoint to these are the sheets of an envelope of spheres whose centers lie on a surface applicable to a paraboloid of revolution, by the theorem of Guichard. Darboux† and Bianchi‡ have considered the pairs of isothermic surfaces which are the envelope of a family of spheres and in conformal correspondence also, and the latter has given the name transformation of Darboux to a transformation of an isothermic surface which changes it into another such that the two form such a pair. Professor Eisenhart considers surfaces with isothermal representation of their lines of curvature and establishes a transformation of such a surface into another of the same kind. His transformation is a transformation of Darboux when the given surface is minimal and only in this case. Moreover, when the surface is minimal, this transformation is the same problem in analysis as the Thybaut transformation. By means of a theorem of Moutard it is shown that with each surface S , whose lines of curvature have isothermal representation on the sphere, there are associated ∞^4 surfaces of the same kind, each of which forms with S the surface envelope of a family of spheres depending on two parameters. The determination of these new surfaces is the same problem apart from quadratures as the finding of the Thybaut transformations of the minimal surface with the same representation of its asymptotic lines as the lines of curvature of S . These transformations admit of a theorem of permutability similar to the theorems by this name shown by Bianchi to be true for Bäcklund and Darboux transformations. In consequence of this theorem, when all the transforms S_1 of a surface S are known, the transforms of the surfaces S_1 can be found without quadrature. Surfaces with plane lines of curvature are of the kind under discussion, as well as certain surfaces with spherical lines of curvature in one system. These

* *Annales de L'Ecole Normale*, ser. 3, vol. 14, pp. 43-98.

† *Annales de L'Ecole Normale*, ser. 3, vol. 16, pp. 465-508.

‡ *Annali di Matem.*, ser. 3, vol. 11, pp. 93-157.

two classes of surfaces and their transformations are studied at length.

4. In central orbits the apsidal angle usually depends upon the apsidal values of the radius vector.* Considering a family of orbits, whose pericentral distance is constant and whose apocentral distance varies, Dr. Griffin discusses the definite integral giving the apsidal angle. He shows that for any force f which is a function of the distance r , such that fr^2 and its derivative with respect to r are both decreasing functions of r , the apsidal angle is a decreasing function of the apocentral distance.

An application is made to the motion of a particle in the equatorial plane of an oblate spheroid, explaining the more rapid "advance of the line of apsides" in the more nearly circular orbits.†

5. While the important part played by the constant of areas h in determining the nature of a central orbit has received some consideration, little attention has been given to the form in which h appears in the solutions. Dr. Griffin's second paper deals with a family of orbits with fixed pericenter, described under a central force, such that the solutions are expressible as power series. The coefficients are shown to be polynomials in $1/h$, having some interesting properties.

The results offer a working basis for the discussion of certain convergence questions. In this connection, as a check, the solutions for the newtonian law are examined also by the method devised by Professor Moulton ‡ for ascertaining the true radii of convergence for related series.

6. It is well known § that orbits described under a central force which is a single-valued function of the distance are symmetric with respect to every apsidal line where the radius vector is a maximum or minimum. Consequently the apsidal angle is constant and the orbit has not more than two apsidal distances. These conclusions, however, may not be drawn when the force is not a single-valued function of the distance.

* Routh, E. J., *Dynamics of a particle*. §§ 368-370.

† Moulton, F. R., *Celestial mechanics*, pp. 232-233.

‡ *Astronomical Journal*, vol. 23, 1903, pp. 93-102.

§ Routh, E. J. *Dynamics of a particle*, pp. 270-272.

In this note, Dr. Griffin considers the ellipse, and shows that with certain interior points as centers of force, the orbit has three distinct apsidal distances and two distinct apsidal angles. The force has the peculiarity that it does not permit a real orbit in all parts of the plane.

7. Professor Miller's first paper aims to exhibit the different ways in which the term commutator has been used in the literature of the group theory, and to point out some of the sources of these disagreements in the hope that these data may tend towards greater uniformity in the use of this term and also make its various meanings less confusing to the reader. It is observed that the definition given by Weber in his *Lehrbuch der Algebra*, volume 2 (1899), page 133, differs from the earlier definition of Dedekind and Frobenius, and that some writers use a definition according to which the commutator is not completely determined by means of its elements. The conclusion reached is that it would be desirable to define $t^{-1}sts^{-1}$ and $s^{-1}tst^{-1}$ as the commutators of s, t and of t, s respectively.

8. Professor Wright shows that if a plane sextic have the maximum number (11) of ovals, they cannot be all external to one another, nor can they lie ten inside the eleventh. His proof consists first in modifying the equation of a sextic supposed to have the given arrangement of ovals so as to produce a sextic with ten isolated points. A further process of modification is next introduced whereby finally a sextic is obtained with eleven isolated points. Such a sextic would be reducible and it is easily seen that it cannot exist. Other arrangements of the ovals are also considered.

9. Professor Osgood's presidential address appears in full in the present number of the BULLETIN.

10. In the BULLETIN for November, 1906, pages 81-84, Dr. Lehmer exhibits a most orderly arrangement for the permutations on n letters. Two important laws are developed for the group matrices of the symmetric, alternating, and metacyclic groups upon n letters and these laws necessitate the development of but one-fourth of the matrix.

In the present paper Miss Schottenfels presents a further study of these particular group matrices, and also diagrams showing the symmetric positions of the matrices of the most

important subgroups with respect to the original ones for $n = 2, 3, 4,$ and 5 .

11. The simplest problem in the calculus of variations is the following : * Given an integral

$$I = \int_{x_0}^{x_1} g(x, y, p)dx,$$

where $p = dy/dx$, to find a curve $y = y(x)$ which joins two given fixed points and renders this integral a minimum. As is well-known, the solutions $y = y(x)$ of the problem must satisfy Euler's † differential equation, which is a differential equation of the second order with a two-parameter family of solutions $y = y(x, \alpha, \beta)$ usually called extremals.

The so-called inverse problem arises when a two-parameter family of curves $y = y(x, \alpha, \beta)$ is given and we seek to determine the integral

$$I = \int_{x_0}^{x_1} g(x, y, p)dx$$

which shall have these curves for its extremals. This inverse problem has been discussed by Darboux, ‡ and it turns out that there are in general an infinite number of solutions.

In Dr. Stromquist's paper a third problem in connection with the integral I is considered, viz., if at every point of the plane a relation between the direction p of an extremal and the direction P transverse to it be given in the form (1) $P = \theta(x, y, p)$, is it then possible to determine an integral I whose extremals and transversals satisfy the given relation? It is shown that an infinitude of functions g can always be determined which will give rise to the relation (1). Further it can be required that a one-parameter family of curves $y = y(x, \gamma)$ be extremals for the integral I . Then it follows that g is uniquely determined provided that in the field composed of extremals $y = y(x, \gamma)$ the value of I as a function of x be given along any curve which is not tangent at any point to a transversal of the field.

In the second part of the paper the above results are applied to some examples.

* Bolza, Lectures on the calculus of variations, § 3.

† Bolza, loc. cit., p. 22.

‡ Darboux, Théorie des surfaces, vol. 3, p. 53. See also Bolza, loc. cit., p. 30.

12. Dr. Richardson's paper is in abstract as follows: Given a series $U(x) = u_1(x) + u_2(x) + \dots$, what is a sufficient condition that integration term by term is permissible? Uniform convergence in a more or less modified form is the condition generally imposed, and the work of Osgood and Arzelà has this for a basis. For large classes of functions this condition is superfluous. Granted that the functions U, u_1, u_2, \dots are integrable, we may without any further condition write down three classes of series which are integrable term by term:

(a) The function u_1, u_2, \dots are all of the same sign ($u_i \geq 0$, $i = 1, 2, 3, \dots$; $u_i \leq 0$).

(b) The series $\int |u_1| + \int |u_2| + \dots$ converges.

(c) For all values of n the sum $\sum_1^n u_i$ is less in absolute value than some arbitrarily large but fixed number.

No restriction is put on the continuity of the functions U, u_1, u_2, \dots , and in classes (a) and (b) these functions may take any values finite or infinite. An extension to functions of m variables is immediate.*

13. The object of Dr. Underhill's paper is the consideration of invariants, under point and parameter transformation, connected with the integrand function of the usual definite integral of the calculus of variations

$$I = \int_{t_0}^{t_1} F(x, y, x', y') dt.$$

The method developed is to start from a known invariant and from this, by a process not essentially different from the S -process of the calculus of variations, to derive new invariants.

Three invariants, denoted by S, K, V , are computed without restricting the arguments x, y , and their derivatives to those of an extremal. In case an extremal is used, a modified form of K is found and connected with it an invariantive normal form of the second variation of the integral I .

Finally S and K are interpreted for the problem of the geodesics on a surface. S is found to be the geodesic curvature; and K , when computed along an extremal, is the gaus-

* Osgood, "Non-uniform convergence and the integration of a series term by term" (*Amer. Journ. of Math.*, vol. 19.). Arzelà, *Sulle serie di funzioni*, Bologna, 1900.

sian curvature. If the value of K is computed along a curve which is not an extremal, it becomes equal to the gaussian curvature minus one half the square of the geodesic curvature.

Incidentally an expression for the Weierstrass function F_2 is found as well as a proof the Jacobi-Bonnet theorem: In the case of the problem of geodesics on a surface of negative curvature, the conjugate point is non-existent.

14. The theory of conservative forces worked out in Professor Kasner's first paper applies to any number of interacting particles, or to the equivalent problem of a single particle in a space of arbitrary dimensionality. The possible motions may be divided into families each characterized by a particular value of the constant of energy. To each value of this constant corresponds a definite system of trajectories, which is termed, following Painlevé, a natural system. The chief object of the paper is to find the geometric properties of such natural systems. The first result states that the circles of curvature constructed at a given point have another point in common. The point-to-point correspondence thus arising is not arbitrary; its characterization is given in the second property. If a system has both properties, it is necessarily natural. The results apply to the related theories of geodesics (Darboux, *Théorie des surfaces*, volume 2, chapter 8) and of contact transformations (Lie, *Berührungstransformationen*, page 102, and *Leipziger Berichte*, 1889).

15. Professor Kasner's second paper deals with certain analogies between the theory of natural systems of dynamical trajectories in the plane, and systems of isogonal trajectories. In both cases the circles of curvature at a given point form a pencil. For isogonals the theorem is due to Cesàro and Scheffers. The types differ however with respect to the nature of the induced point transformation. The author obtains complete characterizations for both types; connects the two theories by a certain transformation of curvature elements; establishes a simple reciprocity between dynamical systems; and finally proves that a natural system (and hence also a geodesic system) is completely determined by an arbitrary simple infinity of curves together with an arbitrary transversal curve.

16. Professor Saurel's paper appears in full in the present number of the BULLETIN.

17. In Professor McKinney's first paper are derived the sufficient conditions in terms of the invariants of the quantic that its roots, supposed distinct, be conecyclic.

18. Professor McKinney's second paper discusses the properties of the continued fractions depending on a variable parameter λ , and representing quadratic irrationalities.

19. The principal theorems proved in Professor Miller's second paper are as follows: If the n different operators s_1, s_2, \dots, s_n are commutative and if each of them is the product of all the others, then they generate the direct product of a cyclic group whose order divides $2(n-2)$ and an abelian group of order 2^α and of type $(1, 1, 1, \dots)$. Moreover, any such direct product may be generated by n operators which satisfy the given condition. If the order of one of these operators is divisible by 4, all of them are of the same order. In fact they always have a common square and the order of each divides $2(n-2)$. The necessary and sufficient condition that two different operators which have a common square are commutative is that one is the product of the other into an operator of order two.

If s_1, s_2, \dots, s_n are not supposed to be commutative but satisfy the condition $s_\alpha = s_{\alpha+1}s_{\alpha+2} \dots s_n s_1 s_2 \dots s_{\alpha-1}$ ($\alpha=1, 2, \dots, n$) they may generate a large number of different types of groups when $n > 4$. In particular, every possible symmetric group whose degree exceeds 3 may be generated by five such operators. All of these operators must have a common square and $s_{n-1}s_{n-2} \dots s_2 s_1 = s_n s_1^{2(n-2)}$. The present paper determines the possible groups for the case when $n=4$, those for smaller values of n being known. It is proved that every group which may be obtained by extending the direct product of two cyclic groups by means of an operator of order two which transforms each operator of this direct product into its inverse may be generated by four operators of order 2, each of which is a product of the other three. A similar theorem is proved for the case when each of the operators s_1, s_2, s_3, s_4 is of order 4. It is observed that each of these four operators is always of even order and that the possible groups are somewhat similar to the dihedral type. In particular, all of them are solvable.

F. N. COLE,
Secretary.