foudent read *fondent;* page 332, last line, letter missing; page 380, for *Thompson* read *Thomson*; page 449, a hyphen is missing in next to the last line, also capitalize d; page 473, for *Veierstrass* read *Weierstrass;* page 481, heading of page is wrong; page 498, 8 lines from bottom; for *snr* read *sur*.

JAMES PIERPONT.

PROJECTIVE DIFFERENTIAL GEOMETRY.

Projective Differential Geometry of Curves and Ruled Surfaces. By E. J. WILCZYNSKI. Leipzig, B. G. Teubner, 1906. viii + 295 pages.

THE present volume is the amplification and systematic development of the ideas originally presented in various papers by the author, published in the *American Journal*, *Transactions of the* AMERICAN MATHEMATICAL SOCIETY and *Mathematische Annalen.*

The work begins with a very brief resumé of the ideas of continuous groups, followed by a synopsis of the transformations of linear homogeneous differential equations, wherein Stäckel's theorem regarding the form of the transformations which leave such an equation invariant is generalized to apply to a simultaneous system of such equations. A fairly full discussion is given to the invariants and covariants of a single linear equation. After showing that every transformation which leaves the equation invariant is of the form

$$y = \lambda(x)\eta, \quad x = f(\xi),$$

the first transformation alone is treated at length, the functions of p_i, $p_i^{(k)}$ which remain invariant being designated as seminvariants; those of p_i, $p_i^{(k)}$, $y^{(l)}$ being called semi-covariants. By the second transformation, a function Ω of x is said to be invariant of weight m if

$$\Omega(x) = \frac{1}{(f')^m}\Omega(\xi).$$

An early application of these ideas is the derivation of the canonical form of the equation, in which the terms containing $y^{(n-1)}$ and $y^{(n-2)}$ are absent. The Lagrange adjoints are discussed and geometric applications made. Every solution of the adjoint furnishes an integrating factor of the equation.

Chapter III contains the first real application of the dominant idea of the book. Given a linear differential equation of order 3

$$(1) \qquad y''' + 3p_1y'' + 3p_2y' + p_3y = 0,$$

the semi-covariants z, ρ are defined by

$$z = y' + p_1y, \quad \rho = y'' + 2p_1y' + p_2y.$$

The one linear invariant is

$$\theta_3 = P_3 - \tfrac{3}{2}P_2',$$

the P_i being the coefficients in the semi-canonical form (*i. e.*, lacking y''). The quadri-derivative of θ_3 (Forsyth) is

$$\theta_{3,1} = 6\theta_3\theta_3'' - 7(\theta_3')^2.$$

Now let three independent solutions of (1) be regarded as homogeneous point coördinates in the plane. As x varies, they will define an analytic curve. The only identically self-dual curves (every tangent associated with its own point of contact) are shown to be conics. The invariants θ_3 and $\theta_{3,1}$, given as arbitrary functions of x, define a plane curve projectively. When the point $(y_1, y_2, y_3) \equiv P_y$ describes a curve, P_z, P_ρ describe other curves, semi-covariantly related to C_y. The equations of the osculating conic (having contact with C_y of the fourth order at a given point) and osculating cubic are now derived. A pencil of C_3's can be found having eight point contact. The residual basis point is called the Halphen point. In particular, it may coincide with the other eight. Such a point is called a coincidence. The cubic now has a double point and no cubic of the pencil has proper contact of the eighth order.

An interesting discussion of C_z, C_ρ follows, including a num ber of anharmonic properties. If all the points of C_y are coincidence points, the curve is projectively equivalent to a logarithmic spiral whose parametric angle is 30°.

If $p_1 = p_2 = 0$, C_3 superosculates C_y. If this happens for all values of x, C_y is itself a C_3. The condition that this happens is expressed in terms of invariants, and a classification of elliptic, nodal, cuspidal cubics is expressed by them. If $\theta_3 = 1$, $\theta_{3,1} = 0$, C_y is C_3.

An application of elliptic functions is then made, showing that the coincidence points on a C_3 are grouped in triangles which are inscribed in and circumscribed about the curve. Finally, the condition that C_y be an anharmonic (or W-) curve is also derived, and various theorems established.

Many of these results were previously obtained by Halphen and others, but the present treatment is much more systematic and complete, and may be read without a knowledge of the previous memoirs on the subject.

These ideas of transformations and invariants of a single differential equation are extended in Chapter IV to a system of linear equations. The first general paper on this subject was published by the author in the *Transactions*, volume 2, pages 1–10. This chapter embodies that paper and pages 99, 104, 106–110, 118–125 are new. Particular attention is paid to two equations, each of the second order, on account of the later application of this system. The equations are reduced to their semi-canonical form and to their canonical form, then the complete system of invariants is derived. The functionally complete system consists of four forms and those resulting from these by repetition of the jacobian process. The system of covariants is also derived. Later a precise geometric meaning is given to each of these forms, and the specializations which appear when one or more of them identically vanishes.

The part of the book that is the essentially novel work of the author commences with Chapter V, foundations of the theory of ruled surfaces. The first paper appeared in the *Transactions*, volume 2, pages 343–362.

From the simultaneous equations of the second order

$$y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z = 0,$$
$$z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z = 0$$

we can define two twisted curves C_y, C_z, by letting x vary. The line (yz) will describe a ruled surface S which is projectively defined by the equations. At the torsal generators of S, the tangents to C_y, C_z intersect. By constructing the tangent planes to S at P_y, P_z we obtain the solutions of another system of linear equations, the adjoint of the given one. The two systems have the same invariants and seminvariants; they are identical if S is a quadric. When the equations are reduced to the semi-canonical form $(p_{12} = p_{21} = 0)$, C_y, C_z are asymptotic lines on S. The fundamental theorem is that when the four basic invariants are given as arbitrary functions of x, not identically zero, then S is uniquely determined, projectively considered. When three basic invariants of one system are equal to those of another, but the fourth negatively equal, the two surfaces are dualistic. If θ_4 is zero, S is identically self-dual, *i. e.*, every point is transformed into the tangent plane at that point and leaves the surface invariant.

The locus of the point through which a transversal can be drawn cutting four consecutive generators of S is the flecnode curve. It is not enveloped by the flecnodal tangents unless it be a straight line directrix. These tangents form another ruled surface, called the flecnode surface of S. As many of these properties are most easily expressed by means of line coordinates, a short resumé of line geometry of the first degree is introduced, then the ideas are applied to interpret the invariants and covariants derived before. Perhaps the most important is that the necessary and sufficient condition that a ruled surface be identically self-dual is that it belong to a non-special linear complex.

If the two branches of the flecnode curve coincide, or if the surface belongs to a special linear complex, it is not determined by means of the four fundamental invariants.

An important idea is that of the flecnode congruence, made up of the generators (of one system) of the osculating hyperboloids H of S. Its focal surface is shown to be the flecnode surface of S and various properties are established. If we put

$$\rho = 2y' + p_{11}y + p_{12}z, \quad \sigma = 2z' + p_{21}y + p_{22}z,$$

then when the line (yz) describes S, the line $(\sigma\rho)$ will describe another surface S', called the derivative of S as to x. It is contained in the flecnode congruence of S, one of its generators lying on the H of each generator of S. Various theorems are established concerning the correspondence between curves on S and on S', both for the general case and when S is contained in one or more linear complexes. If $(yz) \equiv g$, then $(\rho\sigma) \equiv g'$ is a generator of $H(g)$ of the second system, hence $H(g')$, $H(g)$ have g' in common. The residual intersection is therefore in general a space cubic, called the derivative cubic. It cuts g in two points, which with the two flecnodes on g make a harmonic range. Indeed, the idea of harmonic section, first found by Cremona to be made on the generators of certain surfaces by the asymptotic lines, and somewhat extended by other writers, is here shown to permeate the whole theory. Conditions under which the derivative cubic is composite are determined, its linear complex found and a number of properties established. The osculating linear complex of g is introduced to prove and to generalize a number of known theorems on asymptotic lines.

In case H, H' have two lines in common ($\theta_4 = 0$) the residual intersection is the derivative conic. It cannot be composite unless S has a straight line directrix. Two conics belonging to consecutive generators cannot intersect. The developable formed by the plane of the derivative conic is discussed.

We next pass to the discussion of curves upon a surface. An arbitrary curve is one branch of the flecnode curve of an infinite number of ruled surfaces, but two curves chosen at random cannot in general form the complete flecnode curve of any surface. Similarly for the complex curve and for an asymptotic line.

Chapter XIII is concerned with a space curve, defined by a single equation of order 4. Its torsal cubic, osculating cubic, conic and linear complex are treated in detail, and a careful interpretation of the special tetrahedron of reference is given. A detailed comparison with the results of Halphen is then added, with application to anharmonic curves and certain plane curves.

Throughout the book copious references are given, and but few known theorems are left uncredited. A particularly commendable feature is the collection of exercises which follows each chapter; some of them are obvious corollaries of theorems just derived, others are less directly connected, and finally a liberal number indicated by a star are unsolved problems, containing suggestions for further investigation.

The book is provided with an index and is up to the usual standard of excellence maintained by Teubner. Although full of formulas, it is singularly free from typographical errors. Of the fifteen noticed by the reviewer, only two might cause confusion. On page 55, line 22, $M_{n/2}$ should be $M_{n-2/2}$, and on page 67, line 19, Ω_2 should be Ω_1.

VIRGIL SNYDER.

CORNELL UNIVERSITY,
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