THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and twenty-eighth regular meeting of the Society was held in New York City on Saturday, April 28. The following fifty members attended the sessions:

Dr. Grace Andrews, Professor G. A. Bliss, Professor Maxime Bôcher, Professor C. L. Bouton, Professor E. W. Brown, Dr. W. H. Bussey, Professor F. N. Cole, Miss E. B. Cowley, Miss L. D. Cummings, Dr. W. S. Dennett, Professor L. P. Eisenhart, Professor B. F. Finkel, Professor T. S. Fiske, Professor G. H. Hallett, Professor E. R. Hedrick, Dr. L. I. Hewes, Mr. A. M. Hiltebeitel, Professor E. V. Huntington, Mr. S. A. Joffe, Dr. Edward Kasner, Professor C. J. Keyser, Dr. G. H. Ling, Mr. L. L. Locke, Professor E. O. Lovett, Dr. Emory McClintock, Professor James Maclay, Professor H. P. Manning, Professor Max Mason, Mr. A. R. Maxson, Professor W. F. Osgood, Professor James Pierpont, Mr. R. G. D. Richardson, Mr. W. H. Roever, Miss Ida M. Schottenfels, Mr. C. H. Sisam, Dr. Clara E. Smith, Professor D. E. Smith, Professor P. F. Smith, Professor Virgil Snyder, Dr. H. F. Stecker, Dr. C. E. Stromquist, Professor H. D. Thompson, Mr. M. O. Tripp, Professor H. W. Tyler, Professor Oswald Veblen, Professor L. A. Wait, Mr. H. E. Webb, Professor J. B. Webb, Professor H. S. White, Dr. Ruth G. Wood.

President W. F. Osgood occupied the chair. The Council announced the election of the following persons to membership in the Society: Rev. R. D. Carmichael, Hartselle, Ala.; Mr. F. L. Griffin, University of Chicago; Mr. W. R. Longley, University of Chicago; Mr. W. D. MacMillan, University of Chicago; Mr. F. W. Owens, Evanston Academy; Dr. J. J. Quinn, High School, Warren, Pa.; Mr. W. J. Risley, University of Illinois; Dr. R. P. Stephens, Wesleyan University; Mr. J. D. Suter, Iowa State College; Mr. A. M. Wilson, McKinley High School, St. Louis, Mo. Eighteen applications for membership in the Society were received.

Professor W. F. Osgood was elected a member of the Editorial Committee of the *Transactions*, to succeed Professor E. W. Brown, who retires after seven years' service covering the entire period of existence of that journal.

The by-laws were amended to provide that only members of at least four years' standing shall be permitted to compound life membership.

In the interval between the sessions the members lunched together, and the informal dinner in the evening, attended by some thirty members, afforded another welcome opportunity for conference and renewal of acquaintance.

The following papers were read at this meeting:

(1) Professor G. A. MILLER: "Groups in which all the operators are contained in a series of subgroups such that any two of them have only identity in common."

(2) Mr. W. H. ROEVER: "Lines of force illustrated by rotating carriage wheels."

(3) Mr. W. H. ROEVER: "Systems of lines of force whose differential equations take Bernoulli's form in polar coordinates."

(4) Professor VIRGIL SNYDER: "On twisted curves contained in a linear complex."

(5) Mr. G. E. WAHLIN: "On the number of classes of binary quadratic forms and the ideals of a quadratic body."

(6) Mr. R. G. D. RICHARDSON: "On the fundamental theorem in the reduction of multiple integrals."

(7) Professor JAMES PIERPONT: "On the area of curved surfaces."

(8) Professor E. R. HEDRICK: "Functions and their derivatives on given assemblages."

(9) Professor E. R. HEDRICK: "Lipschitz's condition in the case of implicit functions."

(10) Professor MAX MASON: "A necessary condition for an extremum of a double integral."

(11) Professor G. A. BLISS: "An invariant of the calculus of variations corresponding to geodesic curvature."

(12) Dr. EDWARD KASNER: "A generalization of conformal representation."

(13) Dr. EDWARD KASNER: "Velocity curves in the dynamics of a particle."

(14) Professor J. W. YOUNG: "A generalization of a problem of Tchebychev."

(15) Professor C. J. KEYSER: "Concerning the bond uniting elements into a space."

(16) Dr. C. N. HASKINS: "Note on the differential invariants of a plane."

(17) Mr. E. C. COLPITTS: "On twisted quintic curves."

(18) Mr. W. C. BREUKE: "On the differentiation of trigonometric series" (preliminary communication).

(19) Dr. I. E. RABINOVITCH: "The necessary and sufficient kinematic axioms of geometry."

Mr. Wahlin was introduced by Professor Pierpont, Dr. Rabinovitch by Professor Cole; Mr. Colpitts's paper was communicated through Professer Snyder. In the absence of the authors, the papers of Professor Miller and Professor Keyser were read by title, those of Dr. Haskins and Mr. Colpitts were presented by Professor Snyder, and Mr. Breuke's paper by Professor Bôcher. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Professor Miller's paper appears in full in the present number of the BULLETIN.

2. Mr. Roever illustrated, by means of the crossings of spokes of two wagon wheels rotating in parallel planes, the lines of force of a system composed of two sources (of the same or different intensities and of the same or different algebraic signs). The hubs of the wheels correspond to the sources, and the ratio of the angular velocities of the wheels is the negative reciprocal of the ratio of the intensities of the sources. In particular, he considered the case in which a rolling wheel is viewed through a picket fence, and explained why it is that the wheel appears to turn backwards.

3. In this paper Mr. Roever considered the field of force which results from the combination of the fields I and II described below.

Field I. At any point P the direction of the force due to this field passes through a fixed point O, and the magnitude of the force is represented by f_1.

Field II. At any point P the direction of the force due to this field is parallel to the fixed direction OY, and the magnitude is represented by f_2.

If we denote by r the distance OP, and by θ the complement of the angle YOP, the differential equation of the lines of force of the resulting field is

$$(1) \qquad \frac{dr}{d\theta} - \tan\theta \cdot r = -\frac{f_1}{f_2} r \sec\theta.$$

If

$$\frac{f_1}{f_2} = r^n F(\theta),$$

equation (1) takes Bernoulli's form $(dy/dx + P(x) \cdot y = Q(x) \cdot y^n)$.

As applications, Mr. Roever solved the following problems.

(1) Let it be required to find the lines of force of the system which is composed of a uniform electric field of intensity 2κ and an electrified point of charge M. Here

$$f_1 = \frac{m}{r^2}, \quad f_2 = 2\kappa, \quad \frac{f_1}{f_2} = \frac{m}{2\kappa} \frac{1}{r^2}.$$

(2) Let it be required to find the lines of force of the system which is composed of a uniform electric field of intensity 2κ, and an electrified straight line (of infinite length) of charge λ per unit length, the straight line being perpendicular to the lines of force of the uniform field. Here

$$f_1 = \frac{2\lambda}{r}, \quad f_2 = 2\kappa, \quad \frac{f_1}{f_2} = \frac{\lambda}{\kappa} \frac{1}{r}.$$

(3) Assuming the earth to be a rigid sphere with its center of mass at its center of figure, let it be required to find the lines of force of the system which is composed of the gravitational field and the field due to the centrifugal force of rotation. This is the field of force which determines the direction of the plumb line. Here

$$f_1 = \frac{M}{r^2}, \quad f_2 = \omega^2 y = \omega^2 r \sin\theta, \quad \frac{f_1}{f_2} = \frac{M}{\omega^2} \frac{1}{r^3 \sin\theta},$$

where ω is the angular velocity of rotation and M is the mass of the earth.

4. Curves belonging to a linear complex can be depicted into developables containing a conic by means of the Noether point-line transformation. The latter are transformed by duality into curves lying on a quadric cone. In Professor Snyder's paper the details of these transformations are worked out, and a classification of curves contained in a linear complex is given. The simplest such curve of genus one is of order six, of genus two of order seven, of genus five of order eight, etc.

5. Mr. Wahlin presented a simultaneous development of the two theories in question, showing how, by applying the ideal theory to the forms, we are able to simplify the evaluation of the number of classes. He also gave an application of the ideals to the determination of the number of genera of forms.

6. In the *Journal de Mathématiques*, series 4, volume 8 (1892), de la Vallée-Poussin proposes the following problems: Is the existence of the double integral over a plane field T,

$$\int_T f(x, y)dxdy,$$

and of the iterated integral

$$\int_Y dy \int_X f(x, y)dx$$

a sufficient condition for the equality of these integrals? He proves that such is the case if certain restrictions are put on the arrangement of the infinities of the function $f(x, y)$. In the same journal, series 5, volume 5 (1899), he devotes a memoir to this problem, and shows that under certain limitations of the function the theorem is true. He states, however, that in his opinion the theorem is true without these restrictions. Mr. Richardson proved that the existence of these integrals is a sufficient condition for their equality and that the condition of uniform or regular convergence of the integral

$$\int_X f(x, y)dx$$

is entirely unnecessary. It is shown that this theorem is also true for the generalized integrals introduced by Professor Pierpont. Further, if the fields of integration are $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$, of $m + n$, m, and n dimensions respectively, then the existence of

$$\int_{\mathfrak{A}} f(x_1, x_2, \cdots, x_{m+n}) \quad \text{and} \quad \int_{\mathfrak{B}} \int_{\mathfrak{C}} f(x_1, x_2, \cdots, x_{m+n})$$

is a sufficient condition for their equality.

7. Professor Pierpont extended the results of Stolz (*Transactions*, volume 3 (1902), page 23) in a two-fold manner.

First, he considered surfaces whose coordinates do not have first partial derivatives, finite or infinite, at a discrete aggregate of points in the parametric field. Secondly, he considered a much wider system of inscribed polyhedra than is employed by Stolz.

8. In this paper Professor Hedrick discusses the properties of functions of one or more real variables which are defined for the values of the independent variable or variables which constitute a given assemblage. The continuity of such functions, the possibility of adjoining further definition, definitions and theorems concerning continuity, approach to limits, etc., form the first part of the paper.

The discussion of derivatives of such functions leads to an important notion of the derivative of any function on an assemblage, and in particular to the notion of sequence derivatives. After a discussion of these concepts, a general theorem is proved, namely, that if a continuous sequence derivative exists, the ordinary derivative exists and is continuous.

An extension of these results to a generalized notion of derivative and to the concept of a jacobian as an assemblage derivative in space closes the paper.

9. In this paper Professor Hedrick develops the analogon to Lipschitz's condition for the existence of solutions of systems of differential equations, and shows the form which the condition assumes in the case of the solutions of implicit equations. The form of proof given recently by Goursat is used as a basis, but the conditions are materially weakened. In particular the Goursat condition which requires the existence of $\partial F(x, y)/\partial y$ (for an implicit equation $F(x, y) = 0$ which is to be solved for y) is replaced by the weaker condition that the difference quotient

$$Q(x, \bar{y}, \bar{\bar{y}}) \equiv \frac{F(x, \bar{y}) - F(x, \bar{\bar{y}})}{\bar{y} - \bar{\bar{y}}}$$

should lie between two positive numbers (or two negative numbers) for all points near the known point (x_0, y_0) which is on $F(x, y) = 0$. The generalization to n equations is given in several forms.

10. If a function $z(x, y)$ gives an extreme value (maximum or minimum) to the integral

$$J = \int\int f(x, y, z, p, q)\, dxdy \qquad (p = z_x,\ q = z_y),$$

then the first variation of J must vanish, and the second variation must have the same sign for all allowed variations of z. Professor Mason showed that the second variation could be made positive or negative at pleasure, unless

$$f_{pp} f_{qq} - f_{pq}^2 \geqq 0 .$$

This inequality is accordingly a necessary condition for an extremum.

11. The geodesic curvature of a curve on a given surface is usually defined by means of the projection of the given curve on a tangent plane to the surface, but it may also be defined in other ways. For example, if two geodesic lines are drawn tangent to the curve at neighboring points A and A', they will in general general intersect at a point B. The limit as A' approaches A of the ratio of the angle at B between the two geodesics, divided by the length of the arc AA', is the geodesic curvature at the point A.

In the paper of Professor Bliss a generalization of geodesic curvature is found by making use of the definition just given. A problem of the calculus of variations is considered in which the integral has the form

$$I = \int f(x, y, \tau) \sqrt{x'^2 + y'^2}\, dt,$$

where x and y are functions of t defining the given curve, and τ is the angle between the tangent to the curve and the x-axis. In a previous paper the writer has given a generalization corresponding to this integral. If two extremals are taken tangent to the given curve at neighboring points A and A', they will in general intersect, and the generalized angle between them may be determined. The limit of the ratio of this angle to the value of I taken along the arc AA', as A' approaches A, is the desired curvature. It turns out to be an invariant under point transformation.

12. In a previous paper, Dr. Kasner has shown that if a surface is mapped conformally on a plane, the geodesics through any point are pictured by curves whose circles of curvature at the common point form a pencil, the locus of centers of curvature thus being a straight line. In the present paper it is shown that this property belongs to a very general class of representations which includes the conformal class as a special case. The representation is next required to have an additional property, also possessed by conformal representations: the curves on the surface which are depicted by the straight lines $x =$ const., $y =$ const., are to form an orthogonal net. It is shown that non-conformal representations having both properties arise when, and only when, the surface considered is of the Liouville type. The paper concludes with a discussion of possible representations of a plane on itself.

13. Dr. Kasner's second paper relates to the motion of a particle in a plane under the action of any force depending only upon the particle's position. As the particle describes a trajectory its velocity in general varies. For each set of values of x, y, y' v, a unique trajectory is defined, and therefore, at the point in question, a unique value of the curvature or, what is equivalent, of y''. It is thus possible to express the speed v in terms of x, y, y', y''. If now v is replaced by a constant, we have a differential equation of the second order. The curves satisfying this equation are termed "velocity curves." To each speed there corresponds a doubly infinite system of these curves. The first part of the paper investigates these systems, both in general and for conservative, solenoidal, and Laplacian forces. Relations arise to the theory of geodesics, and to isogonal trajectories.

The second part of the paper deals with the total system of ∞^3 velocity curves, corresponding to all values of v. Such a system can never be the system of trajectories either of the same force or of another force. Many of the geometric properties obtained are analogous to the properties of systems of trajectories obtained in a previous paper (see *Transactions* for July, 1906). The properties relate to osculating parabolas and hyperosculating circles; the whole set is completely characteristic.

14. Tchebychev considered the problem of determining, if it exists, the polynomial of given degree n which approximates

most closely to given continuous function of the real variable x in a given finite interval. The generalization considered in Professor Young's paper consists in replacing the given function by an arbitrary function $f(x)$ which is required merely to satisfy a certain functional relation $\phi[f(x)] = 0$ of a general character. The existence of at least one polynomial of (closest) approximation of degree n with reference to ϕ is proved under very general hypotheses on the function ϕ. Under somewhat less general hypotheses, a certain necessary condition is derived that a polynomial of degree n be a polynomial of approximation; and this condition is shown also to be sufficient provided a certain determinant does not vanish. With this same restriction, it is shown that the solution is unique, *i. e.*, that there exists only one polynomial of approximation of degree n with reference to a given ϕ.

15. Professor Keyser's paper considers the question: What, if any, analytic fact corresponds to the universal conviction of "natural" man that the points of (our) space constitute one, a whole—(our) space? The hypothesis is suggested as such that any continuous space ${}_nS_e$ of dimensionality n and of elements e of a kind is constituted a space by virtue of some relation r (different for different S's) subsisting between each e and an entity E (different for different S's) itself not an element of (not *in*) ${}_nS_e$. The hypothesis, naturally not admitting of mathematical demonstration, is nevertheless indicated independently by an infinitude of facts and is not inconsistent with any fact of which the writer is aware. Of such indications the following one may be cited as being at once simple and typical. Denote by S_s the ensemble of spheres orthogonal to a sphere S. The elements of S_s are spheres. Let M be an inhabitant of S_s and be conformed to S_s as the intuition of a human intelligence H is conformed to S_p, ordinary point space. Suppose M and H to write each his own geometry and by chance to employ throughout the same nomenclature. Suppose their works exchanged. Then each may read his own geometry in the other's book. M is certain that his elements (spheres) constitute space. H has precisely the same conviction (feeling) with respect to his elements (points). Neither M nor H perceives the ground of his own conviction. H, however, perceives the (at least a) reason for M's conviction. It is that each of M's elements bears a certain same relation r (orthogonality) to a

certain entity E (sphere S) not in M's space S_s. Reciprocity demands a corresponding perception by M in the case of S_p. Like analogies abound. To an inhabitant of a space, to an intelligence of intuition conformed to it, the space as an entity could not appear, as for example a plane dweller could not behold the plane. In case of S_p, E is probably S_p itself as the latter would appear to a beholder outside; and r probably is distantial; specifically, the distance zero of each point from E, and so of the form

$$(a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5) : \sqrt{a_1^2 + \cdots + a_4^2} = 0.$$

16. The paper of Dr. Haskins calls attention to an error in the recent paper of Professor Forsyth on "Differential invariants of a plane and of curves in the plane." Professor Forsyth states: (1) that there are p^2 algebraically independent absolute invariants of orders not exceeding p; (2) that these invariants are expressible in terms of certain p^2 geometric magnitudes; and (3) that consequently no relation common to all curves in the plane can exist among these geometric magnitudes. — The error in these statements is due to the fact that the differential form under consideration is of class zero, and hence has no gaussian invariants. There are therefore fewer than p^2 absolute invariants, and hence there exist relations among the p^2 geometric magnitudes mentioned. One such relation, of the third order, is readily established.

17. In Mr. Colpitts's paper, quintic curves lying on a hyperboloid were depicted upon a conic, and the bitangents, inflexions, etc., defined in terms of quartic involution of the quadrisecants. The rational form not lying on a quadric was discussed by means of the octic which fixes the points of contact of the stationary planes. The elliptic form was depicted into a plane curve of order 10, and the surface of trisecants was shown to belong to a linear complex. The form of genus 2 was also treated by means of the plane depiction. Finally, an exhaustive classification of the forms belonging to a linear complex was added.

18. In Mr. Breuke's paper the problem of finding the derivative of the function represented by a series of the form

$$(1) \qquad \sum_1^\infty (a_n \cos nx + b_n \sin nx)$$

was considered; and it was shown that if h is a positive integer and α a constant, and if the series

$$(2) \quad \sum_{1-h}^{\infty} \{(n-h)a_{n-h}\cos(nx-h\alpha)-(n+h)a_{n+h}\cos(nx+h\alpha) + (n-h)b_{n-h}\sin(nx-h\alpha)-(n+h)b_{n+h}\sin(nx+h\alpha)\},$$

(where the a's and b's whose subscripts are not positive are all zero) converges uniformly throughout an interval which does not include or reach up to a root of $\sin h(x-a)$, and if (1) converges at some point of this interval, and if as n becomes infinite $\lim a_n = \lim b_n = 0$, then (1) converges uniformly throughout the interval in question, and the function represented by (1) has at every point of this interval a finite derivative which is equal to the value of (2) divided by $2\sin h(x-\alpha)$. If $\alpha=0$, $h=1$, this reduces to a theorem of Lerch. Applications of the theorem to the summation of certain trigonometric series were indicated.

19. The purpose of Dr. Rabinovitch's paper is to lay down a set of necessary and sufficient kinetic axioms of geometry, and to prove their mutual independence and compatibility. The primary elements to start with are bodies and their places, and use is made of an auxiliary postulate of the continuity of time regarded as an aggregate of moments.

The axioms are 7 in number.

1) Divisibility.—Each body is divisible into two bodies.

2) Impenetrability.—Each body or each part occupies a place to the exclusion of other bodies or parts.

3) Motion.— A body can have different places at different times. Motion is proved to be a continuous process depending upon time.

By definition two bodies are contiguous when either can enter in part the region of the other during an infinitesimal interval of time, and a body is termed continuous when each of its parts is contiguous to some other parts of the same body.

Axioms 4) and 5) postulate the existence of continuous bodies termed rigid bodies or solids, *i. e.* such that, 1° any two contiguous parts remain contiguous throughout all their motions, and 2° to each particular place of any given part of such a body there corresponds a unique place for every other given part.

6) Homogeneity.—A part can be separated from a given solid, capable of being wholly placed within the space of any other given solid.

7) Axioms of completeness.—No motion is impossible unless it contradicts the above axioms.

From these are deduced the notions of surface, curve, and point. The places which can be occupied by the same solid, its surface or curve are termed congruent. The sphere, circle, straight line, plane, and angle are deduced from the notion of distance, which is a relation of all congruent couples of points. The socalled parallel postulate is deduced by proving that a certain continuous motion of a figure termed "immaterial quadrilateral" cannot contradict the above axioms.

F. N. COLE,
Secretary.

THE APRIL MEETING OF THE CHICAGO SECTION.

THE nineteenth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at Northwestern University, Evanston, Ill., on Saturday, April 14, 1906. The total attendance was forty, including the following members of the Society:

Dr. L. D. Ames, Mr. G. D. Birkhoff, Professor D. R. Curtiss, Professor E. W. Davis, Professor L. E. Dickson, Dr. E. L. Dodd, Mr. E. B. Escott, Dr. Peter Field, Professor G. W. Greenwood, Professor A. G. Hall, Professor E. R. Hedrick, Professor T. F. Holgate, Mr. Louis Ingold, Professor O. D. Kellogg, Dr. H. G. Keppel, Mr. N. J. Lennes, Professor H. Maschke, Professor E. H. Moore, Professor F. R. Moulton, Dr. L. T. Neikirk, Professor H. L. Rietz, Mr. A. R. Schweitzer, Professor J. B. Shaw, Professor H. E. Slaught, Professor E. J. Townsend, Dr. W. D. Westfall, Professor D. T. Wilson, Mr. R. E. Wilson, Professor Alexander Ziwet.

The chairman of the Section, Professor Alexander Ziwet, presided at the two sessions. Owing to the large number of papers on the programme, it was voted to reduce the time allotments by twenty-five per cent. It was also voted hereafter to restrict the printed programme to those papers for which titles and abstracts are in the hands of the Secretary on the date specified in the preliminary call for the meeting, and to request this notice to be made in the BULLETIN in connection with the announcement of meetings of the Section.