NOTE ON THE HEINE–BOREL THEOREM.

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IN this note is presented a generalized form of the Heine-Borel theorem together with a corollary which is of immediate application in many theorems in metrical analysis. The present form of the theorem is the result of an effort to understand the meaning of the properties of boundedness and unboundedness of sets of points (numbers) in non-metrical analysis situs. It appears that for many purposes the property of boundedness when applied to a closed set may be replaced by the property that the set shall contain its limit points at infinity, *i. e.*, the set shall be closed even if it is unbounded. In non-metrical analysis, therefore, the chief distinction between a closed bounded set and an unbounded set not containing its limit points at infinity seems to be that the latter is necessarily not closed.

The theorem is stated for the case of three dimensions and the language of geometry is used exclusively. In view of the one-to-one correspondence of the set of all points in a three-space and the set of all triples of real numbers, the reader may, if he wishes, regard the geometric language as a notation for a three dimensional number manifold.

§ 1. *Definitions and Preliminary Notions.*

The word region is used to denote any set of points whatever. Two half-lines proceeding from the same point O and not lying in the same line form an angle. (The half-line contains the point O from which it proceeds.) We assume that an angle separates the remaining points of the plane in which it lies into two unique sets, an interior and an exterior set. If the half-lines a, b, c, no two of which lie in the same line and all three of which do not lie in the same plane, proceed from the same point O then the three angles formed by these half-lines together with the interior points of these angles form a trihedron $Oabc$. If A, B, C, are points of the respective half-lines a, b, c, then the four triangles OAB, OBC, OCA, and ABC, together with their interior points, form a tetrahedron. Such tetrahedron we shall speak of as associated with the trihedron $Oabc$. We assume that a trihedron and also a tetrahedron separates space into two unique sets, and hence that these figures have a definite interior.

A finite point P' is a limit point of a set $[P]$ of points if there are points of $[P]$ other than P' within every tetrahedron of which P' is an interior point. If a set $[P]$ of a half-line is unbounded then the set $[P]$ is said to have a limit point ∞ on the half-line. We will regard every half-line as having one point at infinity. A point ∞ on a half-line l proceeding from a point O is said to be a limit point of a set $[P]$ if for every trihedron $Oabc$ containing the points of l as interior points, and for every associated tetrahedron $OABC$ there are points of $[P]$ other than ∞ within $Oabc$ and exterior to $OABC$. A point is within a region R if it is not a limit point of points not of the region. This definition applies to points at infinity, *i. e.*, a point ∞ lies within a region R if there exists a trihedron $Oabc$ and an associated tetrahedron $OABC$ such that ∞ is an interior point of $Oabc$ and further such that every point, other than ∞, within $Oabc$ and exterior to $OABC$ is a point of R. A closed set contains all its limit points. This also applies to points at infinity.

§ 2. *The Generalized Heine-Borel Theorem.*

THEOREM: *If $[P]$ is a closed set of points and if $[R]$ is a set of regions such that every point of $[P]$ is an interior point of at least one region of the set $[R]$ then there exists a finite subset $R_1, R_2, \cdots, R_n$, of regions of the set $[R]$ such that every point of $[P]$ lies within at least one region of the set $R_1, R_2, \cdots, R_n$.*

Proof: Consider any trihedron $Oabc$. Denote by $[P_a]$ the points of $[P]$ which lie on the half-line a. We show first that there is a finite subset of $[R]$, $R_1, R_2, \cdots, R_{n_1}$, such that every point of $[P_a]$ lies within at least one of the regions $R_1, R_2, \cdots, R_{n_1}$. Since the set of all points common to two closed sets is a closed set it follows that the set $[P_a]$ is closed. If the point ∞ of a is a point of $[P_a]$ it follows that there is a region of the set [R] which contains some infinite segment $K\infty$ of a. It remains to show that all points of $[P_a]$ on OK are contained in some finite set of regions of $[R]$. Beginning at O denote by $[P_a']$ the set of all points of $[P_a]$ on OK which are contained within some finite set of regions. This set of points has a least upper bound $\overline{B}[P_a']$, which is a point of

$[P_a]$ since $[P_a]$ is closed. By hypothesis $\bar{B}\ [P_a']$ lies within one region of $[R]$. Then either $\bar{B}[P_a']$ is a limit point of points of $[P_a']$ which are on $\bar{B}[P_a']K$, in which case $\bar{B}[P_a']$ fails to be an upper bound of $[P_a']$, or there is a definite next point of $[P_a]$ on $\bar{B}[P_a']K$ and lying within a region of $[R]$. Hence $B[P_a']$ cannot be an upper bound as just specified. Hence the points of $[P_a]$ on OK which lie within a finite set of regions of the set $[R]$ has no upper bound and therefore there exists a set of regions $R_1, \cdots, R_{n_1}$ of the set $[R]$ such that every point of $[P_a]$ lies within at least one of them.

Obviously there are half-lines k proceeding from O and lying within the angle (a, b) such that every point of $[P]$ which lies on or within the angle (a, b) lies within at least one of the regions $R_1, \cdots, R_{n_1}$. If there does not exist a finite set of regions of the set $[R]$ such that every point of $[P]$ which lies on or within the angle (a, b) lies within at least one of them, then let k' be the bound of all half-lines proceeding from O such that the points of $[P]$ which lie on or within the angle formed by them and a lie within such finite set of regions. By the above proof all points of $[P]$ which lie on k' lie within a finite set of regions of the set [R]. Hence k' cannot be a bound as specified, and every point of $[P]$ which lie on or within the angle (a, b) lies within at least one of a certain finite set $R_1, R_2, \cdots, R_{n_2}$ of regions of the set $[R]$. In the same manner we may show that there exists a subset of $[R]$, $R_1, R_2, \cdots, R_{n_3}$ such that every point of $[P]$ which lies on or within the trihedron $Oabc$ lies with in at least one region of the set $R_1, \cdots, R_{n_3}$. But the three planes determined by the faces of the trihedron $Oabc$ separate the three-space into eight trihedrons of the type $Oabc$. whence the theorem follows for the whole set $[P]$.

This theorem has many immediate corollaries, of which I instance the two following.

1. *Every infinite set of points has at least one limit point which may be finite or infinite.*

2. *If* $z = f(x, y)$ *and if* (a, b) *is a limit point of the set* (x, y) *on which* $f(x, y)$ *is defined then* $z = f(x, y)$ *has some value approached as* (x, y) *approach* (a, b).

Proof: If the theorem fails to hold, then about every point of the line $x = a, y = b$ there is some region within which there is no point of the graph of the function $z = f(x, y)$. By the theorem of this note there is a finite subset of these regions and hence a cylinder enclosing the line $x = a$, $y = b$ within which there is no point of the graph of the function. Hence for values of x and y within this cylinder there are no values of the function, which is contrary to the hypothesis that the function is defined on a set which has (a, b) as a limit point.

§ 3. *A Theorem of Continuity.*

THEOREM: *If $[P]$ is a closed bounded set of points and if $[R]$ is a set of regions such that every point of $[P]$ is an interior point of at least one region of the set then there exists a number d such that a cube whose edge is d and whose center is any point whatever of the set $[P]$ will lie entirely within one of the regions of the set $[R]$.*

Proof: About every point of $[P]$ construct a cube c lying within some region of the set $[R]$. Let the sides of the cubes be parallel to the coördinate planes of a fixed rectangular system. By the Heine-Borel theorem there is a finite subset $c_1, \cdots c_n$ of these cubes such that every point of $[P]$ lies within at least one of them. Obviously, that part of the surface of any cube of $c_1, \cdots, c_n$ which is exterior to all other cubes of the set is made up of a finite number of rectangles. Denote the set of all such rectangles obtained from $c_1, \cdots, c_n$ by $[r]$. No such rectangle contains a point of $[P]$ or is a limit point of $[P]$. Hence there exists a positive number d' which is less than the distance from any point of $[P]$ to any point of $[r]$. Let d'' be a positive number less than the distance from any side s of a cube of $c_1, \cdots, c_n$ to any parallel side (the plane of which does not contain s). Let d''' be the smaller of these two numbers then $d = \frac{1}{2}d'''$ is the required number.

This theorem appears to have been stated in essentially the above form by several persons during the last year. Professor Bolza used it for a set of points in a plane as early as the spring of 1905, and it is possible that the present statement of it is partly due to his suggestion. Mr. Wedderburn has also used the theorem in this form for linear sets. So far as I know it has not been published before now. Dr. Veblen suggested that this form of the theorem be called "the theorem of uniformity." The following is an immediate corollary: "*A function which is continuous over an interval is uniformly continuous over that interval.*"

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