$$N = \frac{1}{m_1}\left[\sum_{\sigma=0}^{(m_1-1)(p_2-2)} P(0, 1, \cdots, p_2 - 2)^{m_1-1}\sigma + \psi\right],$$

where $P(0, 1, \cdots, p_2 - 2)^{m_1-1}\sigma$ stands for the number of partitions of σ in $(m_1 - 1)$'s by the numbers $0, 1, \cdots, p_2 - 2$; and ψ is a determinate function of p_2 and m_1.

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A DEFINITION OF QUATERNIONS BY INDEPENDENT POSTULATES.*

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§ 1. *Quaternions with respect to a Domain D.*†

THE usual theory relates to quaternions $a_1 + a_2 i + a_3 j + a_4 k$ in which the coefficients a_i range independently over all real numbers or else over all complex numbers, and the units have the following multiplication table:

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

These conditions give the real quaternion system and the octonion system.‡ As an obvious generalization, the coefficients may range independently over all the elements of any domain D.

*See Dickson, "On hypercomplex number systems," *Transactions Amer. Math. Society*, vol. 6 (1905).

† A domain consists of any class of elements.

‡ Octonions may be considered as quaternions with complex coefficients.

§ 2. *The Postulates.*

A set of four ordered elements $a = [a_1, a_2, a_3, a_4]$ of D will be called a quaternion a. The symbol $a = [a_1, a_2, a_3, a_4]$ employed is purely positional, without functional connotation. Its definition implies that $a = b$ if and only if $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$, $a_4 = b_4$.

Postulate I. If a and b are any two quaternions, then $s = [a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4]$ is a quaternion.

Definition. Addition of quaternions is defined by $a \oplus b = s$.

Postulate II. $0 = [0, 0, 0, 0]$ is a quaternion.

Postulate III. If 0 is a quaternion, then to any quaternion a corresponds a quaternion a' such that $a \oplus a' = 0$.

Theorem 1. Quaternions form a commutative group under addition.

Postulate IV. a and b being any two quaternions, then $a \oplus b = p = [p_1, p_2, p_3, p_4]$ is a quaternion, where

$$p_1 = a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4, \quad p_3 = a_1b_3 - a_2b_4 + a_3b_1 + a_4b_2$$

$$p_2 = a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3, \quad p_4 = a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1,$$

if the p_i's are in D.

Definition. The product of two quaternions is defined by $a \otimes b = p$.

Theorem 2. Multiplication is not commutative.

Theorem 3. Multiplication is distributive (right and left).

Theorem 4. Multiplication is associative.

To make quaternions four dimensional we add a fifth postulate:

Postulate V. If $\tau_1, \tau_2, \tau_3, \tau_4$ are elements of D such that $\tau_1a_1 + \tau_2a_2 + \tau_3a_3 + \tau_4a_4 = 0$ for every quaternion a, then $\tau_1 = 0$, $\tau_2 = 0$, $\tau_3 = 0$, $\tau_4 = 0$.

Theorem 5. There exist four quaternions $\epsilon_i = [a_{i1}, a_{i2}, a_{i3}, a_{i4}]$ such that $|a_{ij}| \neq 0$.

§ 3. *Identification with Ordinary Quaternions.*

The quaternion system as thus defined is holoedrically isomorphic with the quaternions of Hamilton, the coefficients belonging to the same domain D.

The quaternions $e_1 = [1, 0, 0, 0]$, $e_2 = [0, 1, 0, 0]$, $e_3 = [0, 0, 1, 0]$, $e_4 = [0, 0, 0, 1]$ form a four dimensional system since

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \neq 0.$$

By postulate IV and the definition of multiplication, these quaternions e have the multiplication table

	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	$-e_1$	e_4	$-e_3$
e_3	e_3	$-e_4$	$-e_1$	e_2
e_4	e_4	e_3	$-e_2$	$-e_1$

which, apart from symbolism, is the same as the table of § 1.

§ 4. *On the Independence of the Postulates.*

If D is a domain admitting addition and subtraction, postulates II and III are redundant.

Aside from this case, postulates I–V are independent as shown by the following systems:

(I) Elements 0, [± 1, 0, 0, 0], [0, ± 1, 0, 0], [0, 0, ± 1, 0], [0, 0, 0, ± 1].

(II) D is the domain of positive integers.

(III) Set (II) with 0 added.

(IV) D is the domain of complex numbers, the a_i being pure imaginaries.

(V) a_1 arbitrary; other a's $= 0$.

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