# DETERMINATION OF ASSOCIATED SURFACES.

BY DR. BURKE SMITH.

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It is here proposed to develope a set of formulas for the cartesian coordinates of surfaces which are associate to a given surface referred to its asymptotic lines. The coordinates of an associated surface may thus be found directly from those of the given surface, by differentiation and substitution, when one has solved an equation of the Laplace type.

Since the images on the Gauss sphere of the asymptotic lines on any surface $S_0$ are the same as the images of a conjugate system of lines on the associated surface $S$, we may find $S$ by determining the surface which has, as images of a conjugate system, the lines on the sphere which are the images of the asymptotic lines on $S_0$. Suppose $S_0$ is referred to its asymptotic lines, and let $e, f, g$ represent the fundamental magnitudes of the corresponding sphere. Then in order that the lines $u =$ const., $v =$ const., on the sphere should be the images of the asymptotic lines on $S_0$ it is necessary and sufficient that

$$\frac{\partial}{\partial u}\left\{\begin{matrix}1\,2\\1\end{matrix}\right\}' = \frac{\partial}{\partial v}\left\{\begin{matrix}1\,2\\2\end{matrix}\right\}', \tag{1}$$

where the symbols of Christoffel are formed for the sphere. To find the coordinates of a surface $S$ which has the above lines as images of a conjugate system, we proceed as follows: If $X$, $Y$, $Z$ are the direction cosines of the normals to $S$ along the conjugate lines, the cartesian coordinates $x$, $y$, $z$ of $S$ may be obtained by solving the equations *

$$\begin{cases} xX + yY + zZ = W, \\ x\dfrac{\partial X}{\partial u} + y\dfrac{\partial Y}{\partial u} + z\dfrac{\partial Z}{\partial u} = \dfrac{\partial W}{\partial u}, \\ x\dfrac{\partial X}{\partial v} + y\dfrac{\partial Y}{\partial v} + z\dfrac{\partial Z}{\partial v} = \dfrac{\partial W}{\partial v}, \end{cases} \tag{2}$$

where $W$ is a solution, linearly independent of $X$, $Y$, $Z$, of the equation,

* Bianchi-Lukat, Vorlesungen über Differential-Geometrie, p. 139.

$$(3) \qquad \frac{\partial^2 W}{\partial u \partial v} - \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' \frac{\partial W}{\partial u} - \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' \frac{\partial W}{\partial v} + fW = 0.$$

Now (3) has equal invariants by virtue of (1). If we put $W = \theta \rho^{-\frac{1}{2}}$, where $-\rho^{-2} = K_0$ is the curvature of $S_0$, and take account of Codazzi's equations, which in this case are

$$\frac{\partial \log \rho}{\partial u} = -2 \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}', \quad \frac{\partial \log \rho}{\partial v} = -2 \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}',$$

we may write (3) in the form

$$(4) \qquad \frac{\partial^2 \theta}{\partial u \partial v} \left( \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial u \partial v} - f \right) \theta.$$

$X$, $Y$, $Z$ are particular solutions of (3), and putting $X = \xi / \sqrt{\rho}$, $Y = \eta / \sqrt{\rho}$, $Z = \zeta / \sqrt{\rho}$, $\xi$, $\eta$, $\zeta$ will be particular solutions of (4). Solving (2) for $x$, $y$, $z$ and remembering that $\sqrt{eg - f^2} = [X, \partial Y / \partial u, \partial Z / \partial v]$, we have,

$$(5) \qquad \begin{gathered} x = \frac{[W, \partial Y/\partial u, \partial Z/\partial v]}{\sqrt{eg - f^2}}, \quad y = \frac{[X, \partial W/\partial u, \partial Z/\partial v]}{\sqrt{eg - f^2}}, \\ z = \frac{[X, \partial Y/\partial u, \partial W/\partial v]}{\sqrt{eg - f^2}}. \end{gathered}$$

Now by the formulas of Lelieuvre, the coördinates $x_0$, $y_0$, $z_0$ of $S_0$ satisfy six relations of the form

$$(6) \qquad \frac{\partial x_0}{\partial u} = \zeta \frac{\partial \eta}{\partial u} - \eta \frac{\partial \zeta}{\partial u}, \quad \frac{\partial x_0}{\partial v} = \eta \frac{\partial \zeta}{\partial v} - \zeta \frac{\partial \eta}{\partial v},$$

etc., and since $\rho^2 = \xi^2 + \eta^2 + \zeta^2$ we may write in place of (5)

$$(7) \qquad \left\{ \begin{aligned} x &= \frac{\theta \dfrac{\partial^2 x_0}{\partial u \partial v} - \dfrac{\partial \theta}{\partial u} \dfrac{\partial x_0}{\partial v} - \dfrac{\partial \theta}{\partial v} \dfrac{\partial x_0}{\partial u}}{\rho^{\frac{3}{2}} \sqrt{eg - f^2}} = \frac{\theta \dfrac{\partial^2 x_0}{\partial u \partial v} - \dfrac{\partial \theta}{\partial u} \dfrac{\partial x_0}{\partial v} - \dfrac{\partial \theta}{\partial v} \dfrac{\partial x_0}{\partial u}}{\xi \dfrac{\partial^2 x_0}{\partial u \partial v} - \dfrac{\partial \xi}{\partial u} \dfrac{\partial x_0}{\partial v} - \dfrac{\partial \xi}{\partial v} \dfrac{\partial x_0}{\partial u}} \\ y &= \frac{\theta \dfrac{\partial^2 y_0}{\partial u \partial v} - \dfrac{\partial \theta}{\partial u} \dfrac{\partial y_0}{\partial v} - \dfrac{\partial \theta}{\partial v} \dfrac{\partial y_0}{\partial u}}{\rho^{\frac{3}{2}} \sqrt{eg - f^2}} = \frac{\theta \dfrac{\partial^2 y_0}{\partial u \partial v} - \dfrac{\partial \theta}{\partial u} \dfrac{\partial y_0}{\partial v} - \dfrac{\partial \theta}{\partial v} \dfrac{\partial y_0}{\partial u}}{\eta \dfrac{\partial^2 y_0}{\partial u \partial v} - \dfrac{\partial \eta}{\partial u} \dfrac{\partial y_0}{\partial v} - \dfrac{\partial \eta}{\partial v} \dfrac{\partial y_0}{\partial u}} \end{aligned} \right.$$

$$z = \frac{\theta \frac{\partial^2 z_0}{\partial u \partial v} - \frac{\partial \theta}{\partial u}\frac{\partial z_0}{\partial v} - \frac{\partial \theta}{\partial v}\frac{\partial z_0}{\partial u}}{\rho^{\frac{3}{2}}\sqrt{eg - f^2}} = \frac{\theta \frac{\partial^2 z_0}{\partial u \partial v} - \frac{\partial \theta}{\partial u}\frac{\partial z_0}{\partial v} - \frac{\partial \theta}{\partial v}\frac{\partial z_0}{\partial u}}{\zeta \frac{\partial^2 z_0}{\partial u \partial v} - \frac{\partial \zeta}{\partial u}\frac{\partial z_0}{\partial v} - \frac{\partial \zeta}{\partial v}\frac{\partial z_0}{\partial u}}$$

where $\theta$ is a solution of (4) which is linearly independent of $\xi$, $\eta$, $\zeta$. These equations give the coordinates $x$, $y$, $z$ of the associated surface $S$ from those of $S_0$ when $\theta$ is known.

As an example, consider the sphere. The asymptotic lines on the sphere are its imaginary generatrices, and when referred to these lines its equations take the form

$$x_0 = a\left(\frac{1 + uv}{u + v}\right), \quad iy_0 = a\left(\frac{1 - uv}{u + v}\right), \quad z_0 = a\left(\frac{v - u}{u + v}\right).$$

By direct calculation we have,

$$ds^2 = \frac{-4a^2}{(u + v)^2}du\,dv, \quad K = \frac{1}{a^2}, \quad \rho = ia.$$

Equation (4) becomes in this case, $\partial^2\theta/\partial u \partial v = 2\theta/(u + v)^2$, of which the general solution is

$$\theta = 2(U + V)/(u + v) - U' - V',$$

where $U$ and $V$ are arbitrary functions of $u$ and $v$ respectively and the primes denote derivatives. Substituting in (7) we have, as the equations of the associate of a sphere,

$$x = \tfrac{1}{2}a(U - uU' + \tfrac{1}{2}(u^2 - 1)U'') + \tfrac{1}{2}a(V - vV' + \tfrac{1}{2}(v^2 - 1)V'')$$
$$y = \tfrac{1}{2}ia(U - uU' + \tfrac{1}{2}(u^2 + 1)U'') + \tfrac{1}{2}ia(V - vV' + \tfrac{1}{2}(v^2 + 1)V'')$$
$$z = \tfrac{1}{2}a(uU'' - U') - \tfrac{1}{2}a(vV'' - V').$$

These are the equations of the minimal surfaces.

Consider also the pseudosphere whose equations in terms of the parameters of the asymptotic lines are

$$(8) \qquad \begin{cases} x_0 = i \operatorname{csch}(u + v)\cos(u - v), \\ y_0 = i \operatorname{csch}(u + v)\sin(u - v), \\ z_0 = u + v - \coth(u + v). \end{cases}$$

For the square of the lineal element we have

$$ds^2 = du^2 + 2[\operatorname{csch}^2(u + v) + \coth^2(u + v)]dudv + dv^2$$

and $K = -\rho^{-2} = -1$.

Equation (4) becomes in this case,

$$\partial^2\theta/\partial u\partial v = [\operatorname{csch}^2(u+v) + \coth^2(u+v)]\,\theta.$$

To every solution $\theta$ of this equation, which is linearly independent of $\xi$, $\eta$, $\zeta$, we have from (7)

$$(9)\begin{cases} x = \theta \coth(u+v)\cos(u-v) + \frac{1}{2}(\theta_u + \theta_v)\cos(u-v) \\ \qquad\qquad - \frac{1}{2}(\theta_u - \theta_v)\tanh(u-v)\sin(u-v), \\ y = \theta\coth(u+v)\sin(u-v) + \frac{1}{2}(\theta_u+\theta_v)\sin(u-v) \\ \qquad\qquad + \frac{1}{2}(\theta_u - \theta_v)\tanh(u-v)\cos(u-v), \\ z = i\theta + \frac{1}{2}i(\theta_u+\theta_v)\cos(u+v), \end{cases}$$

where the subscripts denote partial derivatives. From (8) we have $\left\{ {1\,2 \atop 1} \right\} = \left\{ {1\,2 \atop 2} \right\} = 0$. But since $\left\{ {1\,2 \atop 1} \right\} = -\left\{ {1\,2 \atop 1} \right\}'$ and $\left\{ {1\,2 \atop 2} \right\} = -\left\{ {1\,2 \atop 2} \right\}'$ we have for (9), $\left\{ {1\,1 \atop 2} \right\} = 0$ and $\left\{ {2\,2 \atop 1} \right\} = 0$. Hence (9) are the equations of surfaces of Voss, since the parametric lines consist of two families of geodesics which form a conjugate system.*

If the general solution $\theta$ of any equation of the form $\partial^2\theta/\partial u\partial v = M\theta$ is known, where $M$ is any function of $u$ and $v$, and one can find three particular solutions $\xi$, $\eta$, $\zeta$ linearly independent of $\theta$, the equations of a surface $S_0$ referred to its asymptotic lines may be found by quadrature from (6). The surfaces which are associated to $S_0$ may be found by substitution in (7). Thus if one can solve an equation of the above type and can choose the three particular solutions $\xi$, $\eta$, $\zeta$ linearly independent of $\theta$, the cartesian coordinates of pairs of associated surfaces may be found by six quadratures and substitution in (7). As an example, the writer has already considered the equation $\partial^2\theta/\partial u\partial v = 0$ elsewhere.†

Consider also the equation

$$\partial^2\theta/\partial u\partial v = 2\theta/(u+v)^2$$

of which the general solution is

$$\theta = 2(U+V)/(u+v) - U' - V'.$$

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* See Voss, *Sitzungsberichte der Bayrischen Akademie*, 1888, pp. 95–102.

† BULLETIN, Jan. 1906.

As three particular solutions, take

$$\xi = 2U/(u+v) - U', \quad \eta = 2V_1/(u+v) - V_1',$$
$$\zeta = 2V_2/(u+v) - V_2'.$$

Substitution in (6) gives for $S_0$ in this case,

$$x_0 = 2(V_2 V_1' - V_1 V_2')/(u+v) + \int (V_1' V_2'' - V_2' V_1'')dv,$$
$$y_0 = 2(V_2 U' + V_2' U)/(u+v) - U' V_2',$$
$$z_0 = -2(V_1 U' + V_1' U)/(u+v) + U' V_1'.$$

Substitution in (7) gives at once the equations of the group of associated surfaces.

The method above outlined for determining associated surfaces may be applied to the problem of determining surfaces which admit of continuous deformation with preservation of conjugate lines, since such surfaces are the associates of those whose curvature in terms of the parameters of the asymptotic lines is of the form $K = [\phi(u) + \chi(v)]^{-2}$.* In that case (4) may be written in the form

$$\frac{\partial^2\theta}{\partial u \partial v} = \left(-\frac{1}{4\rho^2}\frac{\partial\rho}{\partial u}\frac{\partial\rho}{\partial v} - f\right)\theta.$$

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# NOTE ON THE PRACTICAL APPLICATION OF STURM'S THEOREM.

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IT is strange that the following small point is not mentioned in the text-books. The only remark I can find bearing on it is in Encyklopädie der Mathematik I., 1, 4, page 417. Suppose $f, f_1, \cdots, f_n$ are the successive Sturmian functions in *e. g.*, the case when $f$ has only simple roots. If all the real roots of $f_r$ are known, then we can separate the roots by using $f, f_1, \cdots, f_r$ only. For let $a$, $b$ be two consecutive roots of $f_r$. Then between $a$ and $b$ $f_r$ does not vanish, and therefore no alteration in the number of changes of sign can arise from the functions $f_r, \cdots, f_n$.

* Bianchi-Lukat, l. c., p. 337.