tors in any group in which more than half the operators are of order 2.

A $(2^\alpha, 2^\beta)$ isomorphism between G_1 and the direct product of the dihedral rotation group of order $2^{\beta+1}$ into an operator of order 2 can be established in such a manner as to obtain a group in which the number of operators of order 2 is either $3 + 2^\alpha + 2^{\beta+1} + 2^{\alpha+\beta}$, or $3 + 2^{\alpha+1} + 2^{\beta+1} + 2^{\alpha+\beta-1}, \beta > 0$. In fact, it is possible to form other such isomorphisms, but these two seem especially useful in this connection. Moreover, by establishing a $(2^\alpha, 2^\beta)$ isomorphism between G_1 and a group of order $2^{\beta+2}$ which is constructed in the same way as G_1, we arrive at groups which contain any of the following three numbers of operators of order 2: $3 + 2^\alpha + 2^\beta + 2^{\alpha+\beta}$, $3 + 2^{\alpha+1} + 2^{\beta+1} + 2^{\alpha+\beta-2}$, $3 + 2^{\alpha+1} + 2^\beta + 2^{\alpha+\beta-1}$.

From the above results it follows directly that there are groups of order 2^m which contain any prescribed number of operators of order 2 which satisfies the conditions that it is $\equiv 3$ mod. 4 and less than 124. By other considerations this limit can readily be extended, but my methods seem too special to be given here. It would be interesting to find a number $\equiv 3$ mod. 4 which could not equal the number of operators of order 2 in any group of order 2^m, or to prove the non-existence of such a number.

ON THE ARITHMETIC NATURE OF THE COEFFICIENTS IN GROUPS OF FINITE MONOMIAL LINEAR SUBSTITUTIONS.

BY DR. W. A. MANNING.

(Read before the American Mathematical Society, September 7, 1905.)

PROFESSOR MASCHKE * has proved (with a certain restriction) that the coefficients of finite linear substitution groups can, by proper transformations, be made rational functions of roots of unity. Professor Burnside † has also recently written on this subject. In this note it is proved that linear groups all of

* Maschke, *Math. Annalen* v. 50 (1898), p. 492.
† Burnside, *Proc. London Math. Society*, ser. 2, v. 3 (1905), p. 239.

whose elements are finite monomial substitutions, that is, of the form

$$x_i' = a_j x_j \qquad (i, j = 1, 2, \cdots, n),$$

(n is the number of variables) can be written so that all the coefficients are *roots of unity*.

THEOREM I. *Any non-vanishing element of the principal diagonal of a monomial substitution of finite period is a root of unity.*

Consider the kth power of a substitution S. If a lies in the principal diagonal of S, a^k occupies the same place in the principal diagonal of S^k. Hence a is a root of unity.

THEOREM II. *The product of two non-vanishing elements of the same or any two substitutions of a group* (G) *of finite monomial linear substitutions, which are symmetrically placed with respect to the principal diagonal, is a root of unity.*

The product of these two elements stands in the principal diagonal of the product of the two substitutions.

THEOREM III. *If G has no coefficient a_{ik} zero for all its substitutions, it can be transformed into another monomial group such that the non-vanishing element in the first column of every substitution of G becomes a root of unity.*

Since no element a_{ik} is zero for every substitution of G, we can choose $n - 1$ substitutions

$$A^{(2)} = (a_{ik}^{(2)}), \quad A^{(3)} = (a_{ik}^{(3)}), \cdots, A^{(n)} = (a_{ik}^{(n)}),$$

in which none of the coefficients

$$a_{12}^{(2)}, \ a_{13}^{(3)}, \ \cdots, \ a_{1n}^{(n)}$$

vanish. Transform G by the canonical substitution

$$x_i' = x_i / p_i \qquad (i = 1, 2, \cdots, n).$$

The transformed group (G') is monomial. If p_1 is an arbitrary root of unity, and if $p_2 = a_{12}^{(2)}$, $p_3 = a_{13}^{(3)}$, $\cdots$, $p_n = a_{1n}^{(n)}$, we have in place of $A^{(2)}, A^{(3)}, \cdots, A^{(n)}$ substitutions with p_1 as the only non-vanishing element in the first row of each. Now apply Theorem II to all the substitutions of G', and the present theorem follows.

THEOREM IV. *If G' has no coefficient everywhere zero the non-vanishing elements of every substitution are roots of unity.*

Consider the first column of a product AB. The elements of this column are obtained by multiplying the rows of A into the first column of B. Let A and B run through all the substitutions of G'. Every coefficient is seen to be the quotient of two roots of unity, that is, a root of unity.

Suppose that a certain coefficient a_{ik} vanishes in every substitution of G. We may assume that the variables of G have been so permuted that the $n-r$ last elements in the first row of all the substitutions of G are zero, and that no other row has more than $n-r$ elements which vanish for every substitution of G.

THEOREM V. *Every substitution of G is in the form*

$$\left[\begin{array}{c|c} N_1 & \begin{matrix} 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ \cdot & \cdot & \cdot \end{matrix} \\ \hline \begin{matrix} 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ \cdot & \cdot & \cdot \end{matrix} & N_2 \end{array}\right],$$

where N_1 and N_2 are monomial matrices, without further transformation.

There now are $r-1$ substitutions $A^{(2)}, A^{(3)}, \cdots, A^{(r)}$ in which the coefficients $a_{12}^{(2)}, a_{13}^{(3)}, \cdots, a_{1r}^{(r)}$ do not vanish. From the products

$$A^{(2)}B, A^{(3)}B, \cdots, A^{(r)}B,$$

$$a_{12}^{(2)}b_{2i}=0,\ a_{13}^{(3)}b_{3i}=0,\ \cdots,\ a_{1r}^{(r)}b_{ri}=0 \quad (i=r+1,\cdots,n),$$

where B is any substitution of G. Hence the last $n-r$ coefficients of the first r rows of all the substitutions of G are zero. Since these substitutions are monomial the first r elements in the last $n-r$ rows are also everywhere zero.

The group in the variables $x_1, x_2, \cdots x_r$ has by hypothesis no coefficients that are everywhere zero, so that for it Theorem IV holds.

We continue in this way with the group in the last $n-r$ variables, and finally have the

THEOREM VI. *The coefficients of a group of monomial linear substitutions of finite period may, by means of transformations which leave them monomial, be made roots of unity.*

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