NOTE ON THE STRUCTURE OF HYPER-COMPLEX NUMBER SYSTEMS.

BY DR. SAUL EPSTEEN.

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IN the *Transactions* for April, 1905, on page 176 * the following theorem (No. III.) was enunciated:

Let $E, E_1, E_2, \cdots$ be a normal series of subalgebras of E (i. e., E_r is a maximal invariant subalgebra of E_{r-1}, $E_0 = E$) and let $K_1, K_2, \cdots$ be a series of complementary algebras, such that K_r accompanies E_{r-1} and is complementary to E_r. Under these assumptions the series $K_1\ K_2, \cdots$ is, apart from the order, independent of the choice of the series $E, E_1, E_2, \cdots$. In other words, if $E, E_1', E_2', \cdots$ is any other normal series, the complementary series of algebras $K_1', K_2', \cdots$ which it defines is the same as the series $K_1, K_2, \cdots$, apart from the sequence.

With the exception of this theorem and the corresponding one where a *chief series* is used in place of the *normal series* all the proofs were made in a symbolic notation which is a generalization of one introduced by Frobenius in the theory of abstract groups. † In demonstrating the above proposition we made use of the *characteristic constants* $\gamma_{i_1 i_2 i_3}$ of the system

$$E \equiv e_1 \cdots e_n \qquad (e_{i_1} e_{i_2} = \sum_{i_3} \gamma_{i_1 i_2 i_3} e_{i_3}),$$

which, owing to the associative law, satisfy the n^3 conditions

$$\sum_{i_3} (\gamma_{i_1 i_2 i_3} \gamma_{i_3 i_4 i_5} - \gamma_{i_1 i_3 i_5} \gamma_{i_2 i_4 i_3}) = 0 \quad (i = 1, \cdots, n).$$

In the present note it is shown how this theorem can also be demonstrated in the symbolic notation without recourse to the γ's.

In order to avoid repetitions I employ the same numbering of theorems and equations, and also the same letters and symbols as in the above quoted paper, which this note supplements.

According to the demonstration of the theorem 1, we have (if $E_1 \neq E_1'$)

$$E = E_1 + E_1'.$$

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* Epsteen-Wedderburn, "On the structure of hypercomplex number systems," *Transactions Amer. Math. Society*, vol. 6, pp. 172–178.

† Frobenius, *Berliner Sitzungsberichte*, 1895, p. 164.

Let

$$E_1 \frown E_1' = F_1;$$

then

$$E_1 = F_1 + D_1 \qquad (F_1 \frown D_1 = 0),$$

$$E_1' = F_1 + D_1' \qquad (F_1 \frown D_1' = 0),$$

and it is evident that

$$E = F_1 + D_1 + D_1'.$$

By theorem 2, F_1 is a maximal invariant subalgebra of E_1 and E_1'.

If $F_1, F_2, F_3, \cdots$ is a normal series of F_1, then $E, E_1, F_1, F_2, \cdots$ and $E, E_1', F_1, F_2, \cdots$ are two new normal series of E. For this purpose it is merely necessary to prove that the complementary algebras defined by E, E_1, F_1 and E, E_1', F_1 are the same.

The complementary algebra of $E_1 (= F_1 + D_1)$ with respect to E is

$$(9') \qquad \begin{aligned} E(\text{mod } E_1) &= F_1 + D_1 + D_1' \,(\text{mod } F_1 + D_1) \\ &= D' \,(\text{mod } F_1 + D_1) = D_1' \,(\text{mod } F_1), \end{aligned}$$

since

$$(F_1 + D_1)^2 \leqq F_1 + D_1 \text{ and } D_1'^2 \frown D_1 = 0.$$

Furthermore, the complementary algebra of F_1 with respect to E_1 is

$$(10') \qquad E_1 \,(\text{mod } F_1) = F_1 + D_1 \,(\text{mod } F_1) = D_1 \,(\text{mod } F_1).$$

Now it is seen in the same manner that the complementary algebra of E_1' with respect to E is

$$(10') \qquad D_1 \,(\text{mod } F_1)$$

and that the complementary algebra of F_1 with respect to E_1' is

$$(9') \qquad D_1' \,(\text{mod } F_1)$$

and, therefore, the normal complementary series defined by E, E_1, F_1 is identical, apart from the sequence, with the normal complementary series defined by E, E_1', F_1.

It now becomes necessary to show that the normal series E, E_1, $E_2, \cdots$; E, E_1, F_1, $F_2, \cdots$ define the same complementary series, and similarly for E, E'_1, $E'_2, \cdots$; E, E'_1, F_1, $F_2, \cdots$. This amounts to proving the theorem for the algebras E_1 and E'_1, which involves a finite number of repetitions of the above proof.

A chief or principal series E, P_1, $P_2, \cdots$ of an algebra being defined as one in which P_s is the maximal subalgebra of P_{s-1} which is invariant is $E(P_0 = E)$ it is easily shown in the symbolic notation, by exactly the same process as for the normal series, *that the system of complementary algebras* C_1, $C_2, \cdots$ *is independent of the chief series selected.*

In the case of the normal series the complementary algebras K_1, $K_2, \cdots$ are necessarily simple, but this is not true of the complementary algebras C_1, $C_2, \cdots$ in the case of the chief series.

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A GEOMETRIC PROPERTY OF THE TRAJECTORIES OF DYNAMICS.

BY DR. EDWARD KASNER.

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SUPPOSE that the force acting on a particle whose coordinates are x, y, z, produces an acceleration having the components $\phi\,(x, y, z)$, $\psi\,(x, y, z)$, $\chi\,(x, y, z)$. The equations of motion are then

$$(1) \qquad \ddot{x}=\phi\,(x, y, z),\ \ddot{y}=\psi\,(x, y, z),\ \ddot{z}=\chi\,(x, y, z),$$

where dots denote differentiation with respect to the time t. In such a field of force the initial position and initial velocity completely determine the trajectory. The totality of trajectories thus constitutes a quintuply * infinite system of space curves.

Consider now those trajectories obtained by starting the par-

* The only exception arises in the trivial case where the force is everywhere zero. Then the trajectories are the fourfold infinity of straight lines.