THE HISTORY OF MATHEMATICS IN THE NINETEENTH CENTURY.

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The extraordinary development of mathematics in the last century is quite unparalleled in the long history of this most ancient of sciences. Not only have those branches of mathematics which were taken over from the eighteenth century steadily grown but entirely new ones have sprung up in almost bewildering profusion, and many of these have promptly assumed proportions of vast extent.

As it is obviously impossible to trace in the short time allotted to me the history of mathematics in the nineteenth century, even in merest outline, I shall restrict myself to the consideration of some of its leading theories.

Theory of Functions of a Complex Variable.

Without doubt one of the most characteristic features of mathematics in the last century is the systematic and universal use of the complex variable. Most of the great mathematical theories received invaluable aid from it, and many owe to it their very existence. What would the theory of differential equations or elliptic functions be to-day without it, and is it probable that Poncelet, Steiner, Chasles, and von Staudt would have developed synthetic geometry with such elegance and perfection without its powerful stimulus?

The necessities of elementary algebra kept complex numbers persistently before the eyes of every mathematician. In the eighteenth century the more daring, as Euler and Lagrange, used them sparingly; in general one avoided them when possible. Three events, however, early in the nineteenth century changed the attitude of mathematicians toward this mysterious guest. In 1813–14 Argand published his geometric interpretation of complex numbers. In 1824 came the discovery by Abel of the imaginary period of the elliptic function. Finally Gauss in his second memoir on biquadratic residues (1832) proclaims them a legitimate and necessary element of analysis.

The theory of functions of a complex variable may be said to have had its birth when Cauchy discovered his integral theorem

$$\int_c f(x)dx = 0,$$

published in 1825. In a long series of publications beginning with the Cours d'analyse, 1821, Cauchy gradually developed his theory of functions and applied it to problems of the most diverse nature; e. g., existence theorems for implicit functions and the solutions of certain differential equations, the development of functions in infinite series and products, and the periods of integrals of one-valued and many-valued functions.

Meanwhile Germany is not idle; Weierstrass and Riemann develop Cauchy's theory along two distinct and original paths. Weierstrass starts with an explicit analytic expression, a power series, and defines his function as the totality of its analytical continuations. No appeal is made to geometric intuition, his entire theory is strictly arithmetical. Riemann, growing up under Gauss and Dirichlet, not only relies largely on geometric intuition, but also does not hesitate to impress mathematical physics into his service. Two noteworthy features of his theory are the many-leaved surfaces named after him, and the extensive use of conformal representation.

The history of functions as first developed is largely a theory of algebraic functions and their integrals. A general theory of functions is only slowly evolved. For a long time the methods of Cauchy, Riemann, and Weierstrass were cultivated along distinct lines by their respective pupils. The schools of Cauchy and Riemann were the first to coalesce. The entire rigor which has recently been imparted to their methods has removed all reason for founding, as Weierstrass and his school have urged, the theory of functions on a single algorithm, viz., the power series. We may therefore say that at the close of the century there is only one theory of functions, in which the ideas of its three great creators are harmoniously united.

Let us note briefly some of its lines of advance. Weierstrass early observed that an analytic expression might represent different analytic functions in different regions. Associated with this is the phenomenon of natural boundaries. The question therefore arose as to what is the most general domain of definition of an analytic function. Runge has shown that any connected region may serve this purpose. An important line of investigation relates to the analytic expression of a function by means of infinite series, products, and fractions. Here may be mentioned Weierstrass's discovery of prime factors; the theorems of Mittag-Leffler and Hilbert; Poincaré's uniformization of algebraic and analytic functions by means of a third variable, and the work of Stieltjes, Pincherle, Padé, and Van Vleck on infinite fractions.

Since an analytic function is determined by a single power series, which in general has a finite circle of convergence, two problems present themselves: to determine 1) the singular points of the analytic function so defined, and 2) an analytic expression valid for its whole domain of definition. The celebrated memoir of Hadamard inaugurated a long series of investigations on the first problem; while Mittag-Leffler's star theorem is the most important result yet obtained relating to the second.

Another line of investigation relates to the work of Poincaré, Borel, Stieltjes, and others on divergent series. It is indeed a strange vicissitude of our science that these series, which early in the century were supposed to be banished once and for all from rigorous mathematics, should at its close be knocking at the door for readmission.

Let us finally note an important series of memoirs on integral transcendental functions beginning with Weierstrass, Laguerre, and Poincaré.

Algebraic Functions and Their Integrals.

A branch of the theory of functions has been developed to such an extent that it may be regarded as an independent theory, we mean the theory of algebraic functions and their integrals. The brilliant discoveries of Abel and Jacobi in the elliptic functions from 1824 to 1829 prepared the way for a similar treatment of the hyperelliptic case. Here a difficulty of gravest nature was met. The corresponding integrals have $2p$ linearly independent periods; but, as Jacobi had shown, a one-valued function having more than two periods admits a period as small as we choose. It therefore looked as if the elliptic functions admitted no further generalization. Guided by Abel's theorem, Jacobi at last discovered the solution to the difficulty, 1832; to get functions analogous to the elliptic functions we must consider functions not of one but of p independent variables, viz., the p independent integrals of the first species. The great problem now before mathematicians, known as Jacobi's problem of inversion, was to extend this aperçu to the case of any algebraic configuration and develop the consequences. The first to take up this immense task were Weierstrass and Riemann, whose results belong to the most brilliant achievements of the century. Among the important notions hereby introduced we note the following: the birational transformation, rank of an algebraic configuration, class invariants, prime functions, the theta and multiply periodic functions in several variables. Of great importance is Riemann's method of proving existence theorems as also his representation of algebraic functions by means of integrals of the second species.

A new direction was given to research in this field by Clebsch, who considered the fundamental algebraic configuration as defining a curve. His aim was to bring about a union of Riemann's ideas and the theory of algebraic curves for their mutual benefit. Clebsch's labors were continued by Brill and Noether; in their work the transcendental methods of Riemann are placed quite in the background. More recently Klein and his school have sought to unite the transcendental methods of Riemann with the geometric direction begun by Clebsch, making systematic use of homogeneous coördinates and the invariant theory. Noteworthy also is Klein's use of normal curves in $(p-1)$-way space to represent the given algebraic configuration. Dedekind and Weber, Hensel and Landsberg have made use of the ideal theory with marked success. Many of the difficulties of the older theory, for example the resolution of singularities of the algebraic configuration, are treated with a truly remarkable ease and generality.

In the theory of multiply periodic functions and the general θ functions we mention, besides those of Weierstrass, the researches of Prym, Krazer, Frobenius, Poincaré, and Wirtinger.

Automorphic Functions.

Closely connected with the elliptic functions is a class of functions which has come into great prominence in the last quarter of a century, viz.: the elliptic modular and automorphic functions. Let us consider first the modular functions of which the modulus κ and the absolute invariant J are the simplest types.

The transformation theory of Jacobi gave algebraic relations between such functions in endless number. Hermite, Fuchs, Dedekind and Schwarz are forerunners, but the theory of modular functions as it stands to-day is principally due to Klein and his school. Its goal is briefly stated thus: To determine all subgroups of the linear group

$$x' = \frac{\alpha x + \beta}{\gamma x + \delta}, \tag{1}$$

where $\alpha, \beta, \gamma, \delta$ are integers and $\alpha\delta - \beta\gamma \neq 0$; to determine for each such group associate modular functions and to investigate their relation to one another and especially to J. Important features in this theory are the congruence groups of (1); the fundamental polygon belonging to a given subgroup, and its use as substitute for a Riemann surface; the principle of reflection on a circle; the modular forms.

The theory of automorphic functions is due to Klein and Poincaré. It is a generalization of the modular functions; the coefficients in (1) being any real or imaginary numbers, with non-vanishing determinant, such that the group is discontinuous. Both authors have recourse to non-euclidean geometry to interpret the substitutions (1). Their manner of showing the existence of functions belonging to a given group is quite different. Poincaré by a brilliant stroke of genius actually writes down their arithmetic expressions in terms of his celebrated θ series. Klein employs the existence methods of Riemann. The relation of automorphic functions to differential equations is studied by Poincaré in detail. In particular, he shows that both variables of a linear differential equation with algebraic coëfficients can be expressed uniformly by their means.

Differential Equations.

Let us turn now to another great field of mathematical activity, the theory of differential equations. The introduction of the theory of functions has completely revolutionized this subject. At the beginning of the nineteenth century many important results had indeed been established, particularly by Euler and Lagrange; but the methods employed were artificial, and broad comprehensive principles were lacking. By various devices one tried to express the solution in terms of the elementary functions and quadratures — a vain attempt; for, as we know now, the goal they strove so laboriously to reach was in general unattainable.

A new epoch began with Cauchy, who by means of his new theory of functions first rigorously established the existence of the solution of certain classes of equations in the vicinity of regular points. He also showed that many of the properties of the elliptic functions might be deduced directly from their differential equations. Ere long, the problem of integrating a differential equation changed its base. Instead of seeking to express its solution in terms of the elementary functions and quadratures, one asked what is the nature of the functions defined by a given equation. To answer this question we must first know what are the singular points of the integral function and how it behaves in their vicinity. The number of memoirs on this fundamental and often difficult question is enormous; but this is not strange if we consider the great variety of interesting and important classes of equations which have to be studied.

One of the first to open up this new path was Fuchs, whose classic memoirs (1866–68) gave the theory of linear differential equations its birth. These equations enjoy a property which renders them particularly accessible, viz., the absence of movable singular points. They may, however, possess points of indetermination, to use Fuchs's terminology, and little progress has been made in this case. Noteworthy in this connection is the introduction by von Koch of infinite determinants, whose importance was first shown by our distinguished countryman, Hill; also the use of divergent series — that invention of the devil, as Abel called them — by Poincaré. A particular class of linear differential equations of great importance is the hypergeometric equation; the results obtained by Gauss, Kummer, Riemann, and Schwarz relating to this equation have had the greatest influence on the development of the general theory. The great extent of the theory of linear differential equations may be estimated when we recall that within its borders it embraces not only almost all the elementary functions, but also the modular and automorphic functions.

Too important to pass over in silence is the subject of algebraic differential equations with uniform solutions. The brilliant researches of Painlevé deserve especial mention.

Another field of great importance, especially in mathematical physics, relates to the determination of the solution of differential equations with assigned boundary conditions. The literature of this subject is enormous; we may therefore be pardoned if mention is made only of the investigations of Green, Sturm, Liouville, Bôcher, Riemann, Schwarz, C. Neumann, Poincaré, and Picard.

Since 1870 the theory of differential equations has been greatly advanced by Lie's theory of groups. Assuming that an equation or a system of equations admits one or more infinitesimal transformations, Lie has shown how these may be employed to simplify the problem of integration. In many cases they give us exact information how to conduct the solution and upon what system of auxiliary equations the solution depends. One of the most striking illustrations of this is the theory of ordinary linear differential equations which Picard and Vessiot have developed, analogous to Galois's theory for algebraic equations. An interesting result of this theory is a criterion for the solution of such equations by quadratures. As an application we find that Riccati's equation cannot be solved by quadratures. The attempts to effect such a solution of this celebrated equation in the preceding century were therefore necessarily in vain.

A characteristic feature of Lie's theories is the prominence given to the geometrical aspects of the questions involved. Lie thinks in geometrical images, the analytical formulation comes afterwards. Already Monge had shown how much might be gained by geometrizing the problem of integration. Lie has gone much farther in this direction. Besides employing all the geometrical notions of his predecessors extended to n-way space, he has introduced a variety of new conceptions, chief of which are his surface element and contact transformations.

He has also used with great effect Plücker's line geometry and his own sphere geometry in the study of certain types of partial differential equations of the first and second orders which are of great geometrical interest, for example equations whose characteristic curves are lines of curvature, geodesics, etc. Let us close by remarking that Lie's theories not only afford new and valuable points of view for attacking old problems but also give rise to a host of new ones of great interest and importance.

Groups.

We turn now to the second dominant idea of the century, the group concept.

Groups first became objects of study in algebra when Lagrange 1770, Ruffini 1799, and Abel 1826 employed substitution groups with great advantage to their work on the quintic. The enormous importance of groups in algebra was, however, first made clear by Galois, whose theory of the solution of algebraic equations is one of the great achievements of the century. Its influence has stretched far beyond the narrow bounds of algebra.

With an arbitrary but fixed domain of rationality, Galois observed that every algebraic equation has attached to it a certain group of substitutions. The nature of the auxiliary equations required to solve the given equation is completely revealed by an inspection of this group.

Galois's theory showed the importance of determining the sub-groups of a given substitution group, and this problem was studied by Cauchy, Serret, Mathieu, Kirkman and others. The publication of Jordan's great treatise in 1870 is a noteworthy event. It collects and unifies the results of his predecessors and contains an immense amount of new matter.

A new direction was given to the theory of groups by the introduction by Cayley of abstract groups (1854, 1878). The work of Sylow, Hölder, Frobenius, Burnside, Cole, and Miller deserves especial notice.

Another line of researches relates to the determination of the finite groups in the linear group of any number of variables. These groups are important in the theory of linear differential equations with algebraic solutions; in the study of certain geometrical problems, as the points of inflection of a cubic, the 27 lines on a surface of the third order; in crystallography, etc. They also enter prominently in Klein's Formenproblem. An especially important class of finite linear groups are the congruence groups first considered by Galois. Among the laborers in the field of linear groups we note Jordan, Klein, Moore, Maschke, Dickson, Frobenius, and Wiman.

Up to the present we have considered only groups of finite order. About 1870 entirely new ideas, coming from geometry and differential equations, give the theory of groups an unexpected development. Foremost in this field are Lie and Klein.

Lie discovers and gradually perfects his theory of continuous transformation groups and shows their relations to many different branches of mathematics. In 1872 Klein publishes his Erlanger Programm and in 1877 begins his investigations on elliptic modular functions, in which infinite *discontinuous* groups are of primary importance, as we have already seen. In the now famous Erlanger Programm, Klein asks what is the principle which underlies and unifies the heterogeneous geometric methods then in vogue, as for example, the geometry of the ancients whose figures are rigid and invariable, the modern projective geometry, whose figures are in ceaseless flux, passing from one form to another, the geometries of Plücker and Lie in which the elements of space are no longer points but lines, spheres, or other configurations at pleasure, the geometry of birational transformations, the analysis situs, etc., etc. Klein finds this answer: In each geometry we have a system of objects and a group which transforms these objects one into another. We seek the invariants of this group. In each case it is the abstract group which is essential, and not the concrete objects. The fundamental role of a group in geometrical research is thus made obvious. Its importance in the solution of algebraic equation, in the theory of differential equations, in the automorphic functions we have already seen. The immense theory of algebraic invariants developed by Cayley and Sylvester, Aronhold, Clebsch, Gordan, Hermite, Brioschi, and a host of zealous workers in the middle of the century, also finds its place in the far more general invariant theory of Lie's theory of groups. The same is true of the theory of surfaces as far as it rests on the theory of differential forms. In the theory of numbers, groups have many important applications, for example, in the composition of quadratic forms and the cyclotomic bodies. Finally let us note the relation between hypercomplex numbers and continuous groups discovered by Poincaré.

In resumé, we may thus say that the group concept, hardly noticeable at the beginning of the century, has at its close become one of the fundamental and most fruitful notions in the whole range of our science.

Infinite Aggregates.

Leaving the subject of groups, we consider now briefly another fundamental concept, viz., infinite aggregates. In the most diverse mathematical investigations we are confronted with such aggregates. In geometry the conceptions of a curve, surface, regon, frontier, etc., when examined carefully, lead us to a rich variety of aggregates. In analysis they also appear, for example the domain of definition of an analytical function, the points where a function of a real variable ceases to be continuous or to have a differential coefficient, the points where a series of functions ceases to be uniformly convergent, etc.

To say that an aggregate (not necessarily a point aggregate) is infinite is often an important step; but often again only the first step. To penetrate farther into the problem may require us to state *how* infinite. This requires us to make distinctions in infinite aggregates, to discover fruitful principles of classification, and to investigate the properties of such classes.

The honor of having done this belongs to Georg Cantor. The theory of aggregates is for the most part his creation; it has enriched mathematical science with fundamental and far reaching notions and results.

The theory falls into two parts; a theory of aggregates in general, and a theory of point aggregates. In the theory of point aggregates the notion of limiting points gives rise to important classes of aggregates, as discrete, dense, complete, perfect, connected, etc., which are so important in the function theory.

In the general theory two notions are especially important, viz.: the one-to-one correspondence of the elements of two aggregates, and well ordered aggregates. The first leads to cardinal numbers and the idea of enumerable aggregates, the second to transfinite or ordinal numbers.

Two striking results of Cantor's theory may be mentioned: the algebraic and therefore the rational numbers, although everywhere dense, are enumerable; and secondly, one-way and n-way space have the same cardinal number.

Cantor's theory has already found many applications, especially in the function theory, where it is today an indispensable instrument of research.

Functions of Real Variables. The Critical Movement.

One of the most conspicuous and distinctive features of mathematical thought in the nineteenth century is its critical spirit. Beginning with the calculus, it soon permeates all analysis, and toward the close of the century it overhauls and recasts the foundations of geometry and aspires to further conquests in mechanics and in the immense domains of mathematical physics.

Ushered in with Lagrange and Gauss just at the close of the eighteenth century, the critical movement receives its first decisive impulse from the teachings of Cauchy, who in particular introduces our modern definition of limit and makes it the foundation of the calculus. We must also mention in this connection Abel, Bolzano, and Dirichlet. Especially Abel adopted the reform ideas of Cauchy with enthusiasm and made important contributions in infinite series.

The figure, however, which towers above all others in this movement, whose name has become a synonym of rigor, is Weierstrass. Beginning at the very foundations, he creates an arithmetic of real and complex numbers, assuming the theory of positive integers to be given. The necessity of this is manifest when we recall that until then the simplest properties of radicals and logarithms were utterly devoid of a rigorous foundation; so for example $\sqrt{2}\sqrt{5} = \sqrt{10}$, $\log 2 + \log 5 = \log 10$.

Characteristic of the pre-Weierstrassian era is the loose way in which geometrical and other intuitional ideas were employed in the demonstration of analytic theorems. Even Gauss is open to this criticism. The mathematical world received a great shock when Weierstrass showed them an example of a continuous function without a derivative, and Hankel and Cantor by means of their principle of condensation of singularities could construct analytical expressions for functions having in any interval, however small, an infinity of points of oscillation, an infinity of points in which the differential coefficient is altogether indeterminate, or an infinity of points of discontinuity. Another rude surprise was Cantor's discovery of the one-to-one correspondence between the points of a unit segment and a unit square, followed up by Peano's example of a space filling curve.

These examples and many others made it very clear that the idea of a curve, a surface region, motion, etc., instead of being clear and simple, were extremely vague and complex. Until these notions had been cleared up, their admission in the demonstration of an analytical theorem was therefore not to be tolerated. On a purely arithmetical basis, with no appeal to our intuition, Weierstrass develops his stately theory of functions, which culminates in the theory of abelian and multiply periodic functions.

But the notion of rigor is relative and depends on what we are willing to assume either tacitly or explicitly. As we observed, Gauss, whose rigor was the admiration of his contemporaries, freely admitted geometrical notions. This Weierstrass would criticise. On the other hand Weierstrass has committed a grave oversight: he nowhere shows that his definitions relative to the numbers he introduces do not involve mutual contradictions. If he replied that such contradictions would involve contradictions in the theory of positive integers, one might ask what assurance have we that such contradictions may not actually exist? A flourishing young school of mathematical logic has recently grown up in Italy under the influence of Peano. They have investigated with marked success the foundations of analysis and geometry, and have in particular endeavored to show the non-contradictoriness of the axioms of our number system by making them depend on the axioms of logic, which axioms we must admit in order to reason at all.

The critical spirit which in the first half of the century was to be found in the writings of only a few of the foremost mathematicians has in the last quarter of the century become almost universal, at least in analysis. A searching examination of the foundations of arithmetic and the calculus has brought to light the insufficiency of much of the reasoning formerly considered as conclusive. It became necessary to build up these subjects anew. The theory of irrational numbers invented by Weierstrass has been supplanted by the more flexible theories of Dedekind and Cantor. Stolz has given us a systematic and rigorous treatment of arithmetic. The calculus has been completely overhauled and arithmetized by Thomae, Harnack, Peano, Stolz, Jordan, and Vallée-Poussin.

Leaving the calculus, let us notice briefly the theory of functions of real variables. The line of demarcation between these two subjects is extremely arbitrary. We might properly place in the latter all those finer and deeper questions relating to the number system, the study of the curve, surface, and other geometrical notions, the peculiarities that functions present with reference to discontinuity, oscillation, differentiation, and integration, as well as a very extensive class of investigations whose object is the greatest possible extension of the processes, concepts, and results of the calculus. Among the many not yet mentioned who have made important contributions to this subject we note: Fourier, Riemann, Stokes, Dini, Tannery, Pringsheim, Arzelà, Osgood, Brodén, Ascoli, Borel, Baire, Köpcke, Hölder, Volterra, and Lebesgue.

Closely related with the differential calculus is the calculus of variations; in the former the variables are given infinitesimal variations, in the latter the functions. Developed in a purely formal manner by Jacobi, Hamilton, Clebsch and others in the first part of the century, a new epoch began with Weierstrass who, having subjected the labors of his predecessors to an annihilating criticism, placed the theory on a new and secure foundation, and so opened the path for further research by Schwarz, A. Mayer, Scheeffer, von Escherich, Kneser, Osgood, Bolza, Kobb, Zermelo and others. At the very close of the century Hilbert has given the theory a fresh impulse by the introduction of new and powerful methods which enable us in certain cases to neglect the second variation and to simplify the consideration of the first.

Theory of Numbers. Algebraic Bodies.

The theory of numbers as left by Fermat, Euler and Legendre was for the most part concerned with the solution of diophantine equations, *i. e.*, given an equation $f(x, y, z, \cdots) = 0$, whose coefficients are integers, to find all rational, and especially all integral solutions. In this problem Lagrange had shown the importance of considering the theory of forms. A new era begins with the appearance of Gauss's Disquisitiones Arithmeticæ in 1801. This great work is remarkable for three things; 1) The notion of divisibility in the form of congruences is shown to be an instrument of wonderful power; 2) the diophantine problem is thrown in the background and the theory of forms is given a dominant role; 3) the introduction of algebraic numbers, viz., the roots of unity.

The theory of forms has been further developed along the lines of the Disquisitiones by Dirichlet, Eisenstein, Hermite, H. J. S. Smith, and Minkowski.

Another part of the theory of numbers also goes back to Gauss, viz., algebraic numerical bodies. The law of reciprocity of quadratic residues, one of the gems of the higher arithmetic, was first rigorously proved by Gauss. His attempts to extend this theorem to cubic and biquadratic residues showed that the elegant simplicity which prevailed in quadratic residues was altogether missing in these higher residues until one passed from the domain of real integers to the domain formed of the third and fourth roots of unity. In these domains, as

Gauss remarked, algebraic integers have essentially the same properties as ordinary integers. Further exploration in this new and promising field by Jacobi, Eisenstein, and others soon brought to light the fact that already in the domain formed of the 23d roots of unity the laws of divisibility were altogether different from those of ordinary integers; in particular a number could be expressed as the product of prime factors in more than one way. Further progress in this direction was therefore apparently impossible.

It is Kummer's immortal achievement to have made further progress possible by the invention of his ideals. These he applied to Fermat's celebrated last theorem and the law of reciprocity of higher residues.

The next step in this direction was taken by Dedekind and Kronecker, who developed the ideal theory for any algebraic domain. So arose the theory of algebraic numerical bodies which has come into such prominence in the last decades of the century through the researches of Hensel, Hurwitz, Minkowski, Weber, and above all Hilbert.

Kronecker has gone farther; in his classic Grundzüge he has shown that similar ideas and methods enable us to develop a theory of algebraic bodies in any number of variables. The notion of divisibility, so important in the preceding theories, is generalized by Kronecker still farther in the shape of his system of moduli.

Another noteworthy field of research opened up by Kronecker is the relation between binary quadratic forms with negative determinant and complex multiplication of elliptic functions. H. J. S. Smith, Gierster, Hurwitz, and especially Weber have made important contributions.

A method of great power in certain investigations has been created by Minkowski which he calls the Geometrie der Zahlen. Introducing a generalization of the distance function, he is led to the conception of a fundamental body (Aichkörper). Minkowski shows that every fundamental body is nowhere concave and conversely to each such body belongs a distance function. A theorem of great importance is now the following: The minimum value which each distance function has at the lattice points is not greater than a certain number depending on the function chosen.

We wish finally to mention a line of investigation which makes use of the infinitesimal calculus and even the theory of functions. Here belong the brilliant researches of Dirichlet relating to the number of classes of binary forms for a given determinant, the number of primes in a given arithmetic progression; and Riemann's remarkable memoir on the number of primes in a given interval.

On this analytical side of the theory of numbers we notice also the researches of Mertens, Weber, von Mangoldt, and Hadamard.

Projective Geometry.

The tendencies of the eighteenth century were predominantly analytic. Mathematicians were absorbed for the most part in developing the wonderful instrument of the calculus with its countless applications. Geometry made relatively little progress. A new era begins with Monge. His numerous and valuable contributions to analytic, descriptive, and differential geometry and especially his brilliant and inspiring lectures at the Ecole polytechnique (1795–1809) put fresh life in geometry and prepared it for a new and glorious development in the nineteenth century.

When one passes in review the great achievements which have made the nineteenth century memorable in the annals of our science, certainly projective geometry will occupy a foremost place. Pascal, De la Hire, Monge, and Carnot are forerunners, but Poncelet, a pupil of Monge, is its real creator. The appearance of his Traité des propriétés projectives des figures in 1822 gives modern geometry its birth. In it we find the line at infinity, the introduction of imaginaries, the circular points at infinity, polar reciprocation, a discussion of homology, the systematic use of projection, section, and anharmonic ratio.

While the countrymen of Poncelet, especially Chasles, do not fail to make numerous and valuable contributions to the new geometry, the next great steps in advance are made on German soil. In 1827 Möbius publishes the Barycentrische Calcul; Plücker's Analytisch-geometrische Entwickelungen appears in 1828–31; and Steiner's Systematische Entwickelung der Abhängigkeit geometrischer Gestalten von einander in 1832. In the ten years which embrace the publication of these immortal works of Poncelet, Plücker, and Steiner, geometry has made more real progress than in the 2,000 years which had elapsed since the time of Apollonius. The ideas which had been slowly taking shape since the time of Descartes suddenly crystallized and almost overwhelmed geometry with an abundance of new ideas and principles.

To Möbius we owe the introduction of homogeneous coördinates, and the far reaching conception of geometric transformation including collineation and duality as special cases. To Plücker we owe the use of the abbreviated notation which permits us to study the properties of geometric figures without intervention of the coördinates, the introduction of line and plane coördinates and the notion of generalized space elements. Steiner, who has been called the greatest geometer since Apollonius, besides enriching geometry in countless ways, was the first to employ systematically the method of generating geometric figures by means of projective pencils.

Other noteworthy works belonging to this period are Plücker's System der analytischen Geometrie (1835) and Chasles's classic Aperçu (1837).

Already at this stage we notice a bifurcation in geometric methods. Steiner and Chasles become eloquent champions of the synthetic school of geometry, while Plücker and later Hesse and Cayley are leaders in the analytic movement. The astonishing fruitfulness and beauty of synthetic methods threatened for a short time to drive the analytic school out of existence. The tendency of the synthetic school was to banish more and more metrical methods. In effecting this the anharmonic ratio became constantly more prominent. To define this fundamental ratio without reference to measurement and so to free projective geometry from the galling bondage of metric relations was thus a problem of fundamental importance. The glory of this achievement, which has, as we shall see, a far wider significance, belongs to von Staudt. Another equally important contribution of von Staudt to synthetic geometry is his theory of imaginaries. Poncelet, Steiner, Chasles operate with imaginary elements as if they were real. Their only justification is recourse to the so-called principle of continuity or to some other equally vague principle. Von Staudt gives this theory a rigorous foundation, defining the imaginary points, lines and planes by means of involutions without ordinal elements.

The next great advance made is the advent of the theory of algebraic invariants. Since projective geometry is the study of those properties of geometric figures which remain unaltered by projective transformations, and since the theory of invariants is the study of those forms which remain unaltered (except possibly for a numerical factor) by the group of linear substitutions, these two subjects are inseparably related and in many respects only different aspects of the same thing. It is no wonder then that geometers speedily applied the new theory of invariants to geometrical problems. Among the pioneers in this direction were Cayley, Salmon, Aronhold, Hesse, and especially Clebsch.

Finally we must mention the introduction of the line as a space element. Forerunners are Grassmann, 1844, and Cayley, 1859, but Plücker in his memoirs of 1865 and his work Neue Geometrie des Raumes (1868–69), was the first to show its great value by studying complexes of the first and second order and calling attention to their application to mechanics and optics.

The most important advance over Plücker has been made by Klein who takes as coördinates six line complexes in involution. Klein also observed that line geometry may be regarded as a point geometry on a quadric in five-way space. Other laborers in this field are Clebsch, Reye, Study, Segre, Sturm, and Koenigs.

Differential Geometry.

During the first quarter of the century this important branch of geometry was cultivated chiefly by the French. Monge and his school study with great success the generation of surfaces in various ways, the properties of envelopes, evolutes, lines of curvature, asymptotic lines, skew curves, orthogonal systems, and especially the relation between the surface theory and partial differential equations.

The appearance of Gauss's Disquisitiones generales circa superficies curvas in 1828 marks a new epoch. Its wealth of new ideas has furnished material for countless memoirs and given geometry a new direction. We find here the parametric representation of a surface, the introduction of curvilinear coördinates, the notion of spherical image, the gaussian measure of curvature, and a study of geodesics. But by far the most important contributions that Gauss makes in this work are the consideration of a surface as a flexible inextensible film or membrane and the importance given quadratic differential forms.

We consider now some of the lines along which differential geometry has advanced. The most important is perhaps the theory of differential quadratic forms with their associate invariants and parameters. We mention here Lamé, Beltrami, Mainardi, Codazzi, Christoffel, Weingarten, and Maschke.

An especially beautiful application of this theory is the immense subject of applicability and deformation of surfaces in which Minding, Bauer, Beltrami, Weingarten, and Voss have made important contributions.

Intimately related with the theory of applicability of two surfaces is the theory of surfaces of constant curvature which play so important a part in non-euclidean geometry. We mention here the work of Minding, Bonnet, Beltrami, Dini, Bäcklund, and Lie.

The theory of rectilinear congruences has also been the subject of important researches from the standpoint of differential geometry. First studied by Monge as a system of normals to a surface and then in connection with optics by Malus, Dupin, and Hamilton, the general theory has since been developed by Kummer, Ribaucour, Guichard, Darboux, Voss, and Weingarten. An important application of this theory is the infinitesimal deformation of a surface.

Minimum surfaces have been studied by Monge, Bonnet and Enneper. The subject owes its present extensive development principally to Weierstrass, Riemann, Schwarz, and Lie. In it we find harmoniously united the theory of surfaces, the theory of functions, the calculus of variations, the theory of groups, and mathematical physics.

Another extensive division of differential geometry is the theory of orthogonal systems, of such importance in physics. We note especially the investigations of Dupin, Jacobi, Lamé, Darboux, Combescure, and Bianchi.

We have already mentioned the intimate relation between differential geometry and differential equations developed by Monge and Lie. Among the workers in this fruitful field Darboux deserves especial mention.

One of the most original and interesting contributions to geometry in the last decades of the century is Lie's sphere geometry. As a brilliant application of it to differential geometry we may mention the relation discovered by Lie between asymptotic lines and lines of curvature of a surface. The subject of sphere geometry has been developed also by Darboux, Reye, Laguerre, Loria, P. F. Smith, and E. Müller.

Other Branches of Geometry.

Under this head we group a number of subjects too important to pass over in silence, yet which cannot be considered at length for lack of time.

In the first place is the immense subject of algebraic curves and surfaces. Adequately to develop all the important and elegant properties of curves and surfaces of the second order alone, would require a bulky volume. In this line of ideas would follow curves and surfaces of higher order and class. Their theory is far less complete, but this lack it amply makes good by offering an almost bewildering variety of configurations to classify and explore. No single geometer has contributed more to this subject than Cayley.

A theory of great importance is the geometry on a curve or surface inaugurated by Clebsch in 1863. Expressing the coördinates of a plane cubic by means of elliptic functions and employing their addition theorems, he deduced with hardly any calculation Steiner's theorem relating to the inscribed polygons and various theorems concerning conics touching the curve. Encouraged by such successes Clebsch proposed to make use of Riemann's theory of abelian functions in the study of algebraic curves of any order. The most important result was a new classification of such curves. Instead of the linear transformation Clebsch, in harmony with Riemann's ideas, employs the birational transformation as a principle of classification. From this standpoint we ask what are the properties of algebraic curves which remain invariant for such transformation.

Brill and Noether follow Clebsch. Their method is, however, algebraical and rests on their celebrated residual theorem, which in their hands takes the place of Abel's theorem. We mention further the investigations of Castelnuovo, Weber, Krauss and Segre. An important division of this subject is the theory of correspondences. First studied by Chasles for curves of deficiency 0 in 1864, Cayley and immediately after Brill extended the theory to the case of any p. The most important advance made in later years has been made by Hurwitz, who considers the totality of possible correspondences on an algebraic curve, making use of the abelian integrals of the first species.

Alongside the geometry on a curve is the vastly more difficult and complicated geometry on a surface, or more generally, on any algebraic spread in n-way space. Starting from a remark of Clebsch 1868, Noether made the first great step in his famous memoirs of 1868–74. Further progress has been due to the French and Italian mathematicians. Picard, Poincaré, and Humbert make use of transcendental methods in which figure prominently double integrals which remain finite on the surface and single integrals of total differentials. On the other hand Enriques and Castelnuovo have attacked the subject from a more algebraic-geometric standpoint by means of linear systems of algebraic curves on the surface.

The first invariants of a surface were discovered by Clebsch and Noether; still others have been found by Castelnuovo and Enriques in connection with irregular surfaces.

Leaving this subject let us consider briefly the geometry of n dimensions. A characteristic of nineteenth century mathematics is the generality of its methods and results. When such has been impossible with the elements in hand, fresh ones have been invented; witness the introduction of imaginary numbers in algebra and the function theory, the ideals of Kummer in the theory of numbers, the line and plane at infinity in projective geometry. The benefit that analysis derived from geometry was too great not to tempt mathematicians to free the latter from the narrow limits of three dimensions and so give it the generality that the former has long enjoyed. The first pioneer in this abstract field was Grassmann (1844); we must, however, consider Cayley as the real founder of n-dimensional geometry (1869). Notable contributions have been made by the Italian school, Veronese, Segre, and others.

Non-Euclidean Geometry.

Each century takes over as a heritage from its predecessors a number of problems whose solution previous generations of mathematicians have arduously but vainly sought. It is a signal achievement of the nineteenth century to have triumphed over some of the most celebrated of these problems.

The most ancient of them is the quadrature of the circle, which already appears in our oldest mathematical document, the Papyrus Rhind, B. C. 2000. Its impossibility was finally shown by Lindemann, 1882.

Another famous problem relates to the solution of the quintic, which had engaged the attention of mathematicians since the middle of the sixteenth century. The impossibility of expressing its roots by radicals was finally shown by the youthful Abel (1824), while Hermite and Kronecker (1858) showed how they might be expressed by the elliptic modular functions, and Klein (1875) by means of the icosahedral irrationality.

But of all problems which have come down from the past by far the most celebrated and important relates to Euclid's parallel axiom. Its solution has profoundly affected our views of space and given rise to questions even deeper and more far reaching, which embrace the entire foundation of geometry and our space conception. Let us pass in rapid review the principal events of this great movement. Wallis in the seventeenth, Saccheri, Lambert, and Legendre in the eighteenth are the first to make any noteworthy progress before the nineteenth century. The really profound investigations of Saccheri and Lambert strangely enough were entirely overlooked by later writers and have only recently come to light.

In the nineteenth century non-euclidean geometry develops along four directions which roughly follow each other chronologically. Let us consider them in order.

The naive-synthetic direction. — The methods employed are similar to those of Euclid. His axioms are assumed with the exception of the parallel axiom; the resulting geometry is what is now called hyperbolic or Lobachevsky's geometry. Its principal properties are deduced; in particular its trigonometry, which is shown to be that of a sphere with imaginary radius, as Lambert had divined. As a specific result of these investigations the long-debated question relating to the independence of the parallel axiom was finally settled. The great names in this group are Lobachevsky, Bolyai, and Gauss. The first publications of Lobachevsky are his Exposition succincte des principes de la géométrie (1829) and the Geometrische Untersuchungen (1840). Bolyai's Appendix was published in 1832. As to the extent of Gauss's investigations we can only judge from scattered remarks in private letters and his reviews of books relating to the parallel axiom. His dread of the Geschrei der Böotier, i. e., the followers of Kant, prevented him from publishing his extensive speculations.

The metric-differential direction. — This is inaugurated by three great memoirs by Riemann, Helmholtz, and Beltrami, all published in the same year, 1868.

Beltrami, making use of results of Gauss and Minding, relating to the applicability of two surfaces, shows that the hyperbolic geometry of a plane may be interpreted on a surface of constant negative curvature, the pseudosphere. By means of this discovery the purely logical and hypothetical system of Lobachevsky and Bolyai takes on a form as concrete and tangible as the geometry of a plane.

The work of Riemann is as original as profound. He considers space as an n-dimensional continuous numerical multiplicity which is distinguished from the infinity of other such multiplicities by certain well-defined characters. Chief of them are, 1) the quadratic differential expression which defines the length of an element of arc, and 2) a property relative to the displacements of this multiplicity about a point. There are an infinity of space multiplicities which satisfy Riemann's axioms. By extending Gauss's definition of the curvature k of a surface at a point to curvature of space at a point by considering the geodesic surfaces passing through that point, Riemann finds that all these spaces fall into three classes according as k is equal to, greater, or less than 0. For $n = 3$ and $k = 0$ we have euclidean space; when $k < 0$ we have the space found by Gauss, Lobachevsky and Bolyai; when $k > 0$ we have the space first considered in the long forgotten writings of Saccheri and Lambert, in which the right line is finite.

Helmholtz like Riemann considers space as a numerical multiplicity. To further characterize it Helmholtz makes use of the notions of rigid bodies and free mobility. His work has been revised and materially extended by Lie from the standpoint of the theory of groups.

In the present category, as also in the following one, belong important papers by Killing.

The projective direction. — We have already noticed the efforts of the synthetic school to express metric properties by means of projective relations. In this the circular points at infinity were especially serviceable. An immense step in this direction was taken by Laguerre who showed, 1853, that all angles might be expressed as anharmonic ratios with reference to these points, i. e., with reference to a certain fixed conic. The next advance is made by Cayley in his famous sixth memoir on quantics, 1859. Taking any fixed conic (or quadric, for space) which he calls the absolute, Cayley introduces two expressions depending on the anharmonic ratio with reference to the absolute. When this degenerates into the circular points at infinity, these expressions go over into the ordinary expressions for the distance between two points and the angle between two lines. Thus all metric relations may be considered as projective relations with respect to the absolute. Cayley does not seem to be aware of the relation of his work to non-euclidean geometry. This was discovered by Klein (1871). In fact, according to the nature of the absolute various geometries are possible; among these are precisely the three already mentioned. Klein has made many important contributions to non-euclidean geometry. We mention his modification of von Staudt's definition of anharmonic ratio so as to be independent of the parallel axiom, his discovery of the two forms of Riemann's space, and finally his contributions to a class of geometries first noticed by Clifford, which are characterized by the fact that only certain of its possible displacements can affect space as a whole.

As a result of all these investigations both in the projective, as also in the metric differential direction we are led irresistibly to the same conclusion, viz.: The facts of experience can be explained by all three geometries when the constant k is taken small enough. It is therefore merely a question of convenience whether we adopt the parabolic, hyperbolic, or elliptic geometry.

The critical synthetic direction represents a return to the old synthetic methods of Euclid, Lobachevsky and Bolyai with the added feature of a refined and exacting logic. Its principal study is not of non-euclidean but of euclidean geometry. Its aim is to establish a system of axioms for our ordinary space which are complete, compatible, and irreducible. The fundamental terms, point, line, plane, between, congruent, etc., are introduced as abstract marks whose properties are determined by inter-relations in the form of axioms. Geometric intuition has no place in this order of ideas which regards geometry as a mere division of pure logic. The efforts of this school have already been crowned with eminent success, and much may be expected from it in the future. Its leaders are Peano, Veronese, Pieri, Padoa, Burali-Forti, and Levi-Civita in Italy, Hilbert in Germany, Moore in America, and Russell in England.

Closing at this point our hasty and imperfect survey of mathematics in the last century let us endeavor to sum up its main characteristics. What strikes us at once is its colossal proportions and rapid growth in nearly all directions, the great variety of its branches, the generality and complexity of its methods, an inexhaustible creative imagination, the fearless introduction and employment of ideal elements, and an appreciation for a refined and logical development of all its parts.

We who stand on the threshold of a new century can look back on an era of unparalleled progress. Looking into the future an equally bright prospect greets our eyes; on all sides fruitful fields of research invite our labor and promise easy and rich returns. Surely this is the golden age of mathematics!

OUTER ISLAND,
September, 1904.

DE SÉGUIER'S THEORY OF ABSTRACT GROUPS.

Eléments de la Théorie des Groupes Abstraits. By J.-A. DE SÉGUIER. Paris, Gauthier-Villars, 1904. ii + 176 pp.

THE title for the complete treatise is Théorie des groupes finis. The present first volume deals with the theory as far as it demands no concrete representation. The second volume is to be entitled Compléments.

The Eléments gives a remarkably compact presentation of purely abstract group theory, including the most recent results. The attempt has been made to extend as far as possible the general theorems to infinite groups. The broader view thus gained more than compensates for the increased abstruseness. It appeals particularly to the reviewer who has given much attention to the coördination of the various branches of analytic group theory into a comprehensive theory of analytic groups in an arbitrary field. The inclusion of infinite groups, moreover, gives the author the means of a natural presentation of negative and rational numbers, Galois's imaginaries, and algebraic numbers as elements of certain groups. The author is therefore justified in giving (pages 27–51) a very compact, but practically complete, account of Galois fields (champ, corps de Galois). Relative to a first mode of composition, called addition, C_N is an additive group; relative to a second mode of composition C_N, with zero omitted, is a multiplicative group, and one may set $0x = x0 = 0$ by definition; a final postulate makes multiplication distributive with respect to addition.

The opening six pages on Cantor's assemblages establish his distinction between finite and infinite sets, but make no classification of the latter. Throughout the text the term *corps* is