THE DETERMINATION OF THE CONSTANTS IN THE PROBLEM OF THE BRACHISTOCHRONE.*

BY PROFESSOR OSKAR BOLZA.

THE general solution of Euler's differential equation for the problem of the brachistochrone in a vertical plane is the doubly infinite system of cycloids †

$$\begin{aligned} x - x_0 + h &= \pm r(\omega - \sin \omega), \\ y - y_0 + k &= r(1 - \cos \omega), \end{aligned} \tag{1}$$

referred to a rectangular system of coördinates whose (x, y)-plane is the given vertical plane and whose positive y-axis is directed vertically downward; x_0, y_0 are the coördinates of the starting point A, k is a given constant, viz.,

$$k = v_0^2/2g,$$

where v_0 is the initial velocity and g the constant of gravity; finally h and r are the two constants of integration, and r is essentially positive.

We suppose that the endpoints A and B are fixed and propose *to determine the constants of integration so that the cycloid*

*Very little attention seems to have been paid to the question of the determination of the constants in the problem of the brachistochrone. I have been able to find only the following few references on the subject in the literature of the calculus of variations :

Johann Bernoulli who first proposed the problem of the brachistochrone in 1696, and Jacob Bernoulli give a geometrical construction for the special case where the initial velocity is zero, in which case the starting point is a cusp of the cycloid (compare Ostwald's Klassiker der exacten Wissenschaften, no. 46, pp. 12 and 18).

Dienger, Grundriss der Variationsrechnung (1867), p. 38, reduces for the general case the determination of the constants to the two transcendental equations (7) and (8) of the text without entering into a further discussion of these equations.

Weierstrass in his lectures (1882) states without proof that it is always possible to construct a cycloid upon a given base passing through two given points A and B, and only one cycloid which contains no cusp between A and B.

† Compare for instance Lindelöf-Moigno, Calcul des variations, p. 228, and Pascal, Die Variationsrechnung, § 31, where numerous historical references are given.

*shall pass through the two given points A and B, with the additional condition that no cusp * of the cycloid shall lie between A and B and that the angle ω increases as the point x, y describes the cycloid from A to B.*

We shall prove that this problem has always *one and but one solution* provided that

$$x_1 \neq x_0 \quad \text{and} \quad y_1 - y_0 + k \geqq 0. \tag{2}$$

If we denote by x_1, y_1 the coördinates of the endpoint B and by ω_0 and ω_1 the values of the angle ω at the two points A and B respectively, we have to determine the four quantities h, r, ω_0, ω_1, from the four equations

$$\begin{aligned} h &= \pm r(\omega_0 - \sin \omega_0), & x_1 - x_0 + h &= \pm r(\omega_1 - \sin \omega_1), \\ k &= r(1 - \cos \omega_0), & y_1 - y_0 + k &= r(1 - \cos \omega_1), \end{aligned} \tag{3}$$

with the additional inequality †

$$0 \leqq \omega_0 < \omega_1 \leqq 2\pi. \tag{4}$$

Since the function $\omega - \sin \omega$ always increases as ω increases, the upper sign must be taken when $x_1 > x_0$, the lower sign when $x_1 < x_0$.

Leaving aside the two special cases

$$v_0 = 0, \quad \text{and} \quad v_0 \neq 0, \quad y_1 = y_0,$$

in which the solution is rather trivial, we assume, in addition to the inequalities (2), that $v_0 \neq 0$ and therefore

$$k > 0, \tag{5}$$

and that

$$y_1 \neq y_0. \tag{6}$$

If we eliminate h and r between the four equations (3), we obtain

$$m[\omega_1 - \omega_0 - (\sin \omega_1 - \sin \omega_0)] = \cos \omega_0 - \cos \omega_1, \tag{7}$$

$$a \sin \frac{\omega_0}{2} = \sin \frac{\omega_1}{2}, \tag{8}$$

* An arc of a cycloid containing a cusp in its interior can never furnish a minimum for the time (Schwarz, Lectures, 1898–99, unpublished).

† If ω_0, ω_1, h, r is a solution of (4) and if μ is an integer, then $\omega_0 + 2\mu\pi$, $\omega_1 + 2\mu\pi$, $h \pm 2\mu\pi r$, r is also a solution, furnishing however the same cycloid. Hence we may restrict ω_0 and ω_1, to the above interval without loss of generality.

where

$$m = \pm \frac{y_1 - y_0}{x_1 - x_0},$$

and

$$a = \sqrt{\frac{y_1 - y_0 + k}{k}},$$

the square root being taken positive.

According as $y_1 \gtrless y_0$, we have $m \gtrless 0$ and $a \gtrless 1$.

If we set

$$u = \frac{\omega_0 + \omega_1}{2}, \qquad v = \frac{\omega_1 - \omega_0}{2}, \tag{9}$$

we obtain from (8)

$$tg\frac{u}{2} = \frac{a+1}{a-1} tg\frac{v}{2}. \tag{10}$$

Hence if we introduce u and v in (7), express u in terms of v by means of (10) and put for brevity

$$z = tg\frac{v}{2}, \quad c = \frac{a-1}{a+1}, \quad b = \frac{c}{m}, \tag{11}$$

the problem reduces to the solution of the transcendental equation

$$F(z) \equiv \operatorname{arc} tg z - \frac{z(c^2 + 2bz - z^2)}{(1+z^2)(c^2+z^2)} = 0. \tag{12}$$

On account of (4), the angle

$$\frac{v}{2} = \operatorname{arc} tg\, z$$

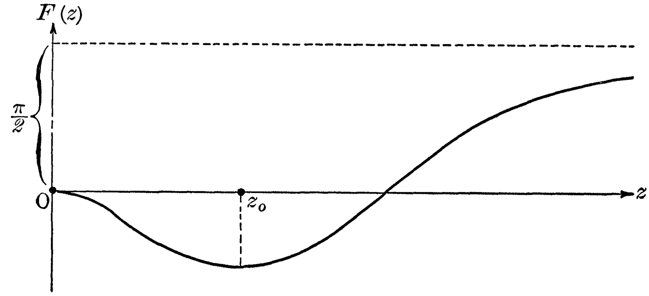
lies between 0 and $\pi/2$ and therefore z must be positive.

For the derivative of $F(z)$ we obtain

$$F'(z) = \frac{z[4bz^4 + 2(1+3c^2)z^3 + 2c^2(3+c^2)z - 4bc^2]}{(z^2+1)^2(z^2+c^2)^2}.$$

Since we suppose $x_0 \neq x_1$ and $y_0 \neq y_1$, m and c are both finite and different from zero and have the same sign, viz., the same sign as $y_1 - y_0$. Hence b is always positive. It follows therefore from Descartes's rule that the numerator of $F'(z)$ has *one and but one real positive root,* say $z = z_0$. Since moreover $F'(z)$

is negative for sufficiently small positive values of z, it follows that $F(z)$ decreases from the value zero to a certain negative minimum value, as z increases from 0 to z_0; then as z increases



from z_0 to $+\infty$, $F(z)$ increases continually and approaches for $z = +\infty$ the limit $+\pi/2$. Hence the equation (12) has one and but one real positive root.

This root being found, the equations (11), (10), and (9) yield a unique solution ω_0, ω_1, of (7) and (8), satisfying the inequality (4). Finally the values of h are z follow unambiguously from (3).

The existence and uniqueness of the solution of the proposed problem are therefore proved.

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ON THREE TYPES OF SURFACES OF THE THIRD ORDER REGARDED AS DOUBLE SURFACES OF TRANSLATION.

BY DR. A. S. GALE.

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THIS note serves the double purpose of making a slight addition to the theory of three types of surfaces of the third order and of exhibiting the double surfaces of translation of lowest order. The latter surfaces enjoy all the properties of the double minimum surfaces * except those immediately dependent on the

* Lie, *Math. Annalen*, vol. 14 (1879), p. 346 et seq.; Darboux, Théorie des surfaces, vol. 1, p. 348 et seq.