$$+ \cdots + \alpha'^{\phi(P)}\pi^{\phi(P)} \equiv 0, \text{ mod. } P^{n+1},$$

where $\alpha\alpha' \equiv 1$, mod. P^n.

If $m < n$, all the terms in (8) would be divisible by P^m, and hence $\phi(P)$ divisible by P, which is impossible. Hence we must have $m = n$. Then we get from (8)

$$\pi \equiv \alpha\mu_n q_n(\alpha), \text{ mod. } P^{n+1},$$

and

$$(9) \qquad \beta \equiv \alpha[1 + \mu_n q_n(\alpha)], \text{ mod. } P^{n+1}.$$

It is also easily seen that $\alpha[1 + \mu_n q_n(\alpha)]$ is a root of (7), if α is a root of (6). Now let α_1 and α_2 be two roots of (6), incongruent mod. P^n. Then, if

$$\alpha_1[1 + \mu_n q_n(\alpha_1)] \equiv \alpha_2[1 + \mu_n q_n(\alpha_2)], \text{ mod. } P^{n+1},$$

we should have

$$\alpha_1 - \alpha_2 \equiv \mu_n[\alpha_2 q_n(\alpha_2) - \alpha_1\mu_n q_n(\alpha_1)], \text{ mod. } P^{n+1},$$

which is impossible, since $\alpha_1 - \alpha_2$ is not divisible by P^n.

Now by giving to n the values 1, 2, 3, $\cdots$ we thus see that all the roots of

$$x^{\phi(P)} \equiv 1, \text{ mod. } P^n,$$

are

$$(10) \quad x \equiv \alpha[1 + \mu_1 q_1(\alpha)] \cdots [1 + \mu_{n-1}q_{n-1}(\alpha)], \text{ mod. } P^n,$$

where α runs through the roots of

$$x^{\phi(P)} \equiv 1, \text{ mod. } P.$$

PURDUE UNIVERSITY,
August, 1903.

MACH'S MECHANICS.

The Science of Mechanics — a Critical and Historical Account of its Development. By ERNST MACH. Translated from the German by T. J. MCCORMACK. Second revised and enlarged edition. Chicago, The Open Court Publishing Co., 1902. xix + 605 pp.

IN a recent review of the German edition of Routh's Rigid Dynamics, BULLETIN, May, 1902, we expressed the desire that English and American publishers were as willing to render the great works of foreign scientists into English as Teubner is to render them into German. The Open Court Publishing Company has undertaken the task to a certain extent in publishing translations of Hilbert's Foundations of Geometry, Dedekind's Essays on the Theory of Numbers, and several volumes of Professor Mach's works. For this our thanks are certainly due and we can cordially rejoice that the pains taken have been rewarded by the demand for a second edition of The Science of Mechanics in less than ten years after its first English publication.

The edition, whether German or English, now before the public is practically definitive: for in the preface the author states: "I desire also that no changes shall be made in it even if after my death a new edition should become necessary." When the remarkable fact is borne in mind that, although many different points of view in treating science have appeared in the last quarter of a century, the main part of Professor Mach's text has remained practically unchanged since its first edition in 1893 and that the alterations have been in the form of appendices added to defend the text or to explain its connection with other works which have followed a similar line of ideas, one may well believe that what Professor Mach himself has been unable to better, future editors would not have had the assurance to alter even if he had not expressed himself as averse to such possible changes.

Before passing to the technical discussion of the volume it might be well to note a few of the words of wisdom — wisdom of the inspired yet common sense kind — which are sprinkled into the text so thickly that they alone repay a perusal of the book. "Apart from the consideration that we cannot afford to neglect the great incentives that it is in our power to derive from the foremost intellects of all epochs — incentives which taken as a whole are more fruitful than the greatest men of the present day are able to offer — there is no grander, no more intellectually elevating spectacle than the utterances of the fundamental investigators in their gigantic power. * * * In fact the mania for demonstration in science results in a rigor that is false and mistaken. Some propositions are held to be possessed of more certainty than others and even regarded as their necessary and incontestable foundation; whereas actually no higher, or perhaps not even so high, a degree of certainty attaches to them." The deductions of Archimedes, not considering their historical value, are infected with this erroneous vigor. But the most conspicuous example of all is furnished by Daniel Bernoulli's deduction of the parallelogram of forces. * * * The historical investigation of the development of a science is most needful, lest the principles treasured up in it become a system of half-understood prescripts, or worse, a system of *prejudices*. * * * Science itself therefore may be regarded as a minimal problem, consisting of the completest possible presentment of facts with the least possible expenditure of thought."

By the word mechanics in the title Professor Mach means in reality what is now-a-days only a very small part of mechanics. He means the statics and kinetics of a particle or system of particles and the elements (historical) of hydrostatics and hydrokinetics. There is no passage from a number of particles to a rigid body. Consequently there is no mention of the theories of elasticity. Even a number of important researches of a time now long passed, such for example as Euler's equations of motion of a fluid, are not mentioned. It is merely the most elementary and fundamental things that receive attention. In general the purely mathematical developments are cut down to the smallest number possible. So the book may be read without any very extended knowledge of differential and integral calculus. Yet, as was evidently advisable, the author goes carefully into the employment of the calculus of variations by Lagrange and into the minimal properties of Maupertuis, Gauss and Hamilton. The treatment of all these matters has, however, been rendered as easy as possible and is most strongly to be recommended to all who have found difficulty in other presentations.

It would not be too much to state in general that for those subjects which the book treats, it may serve not merely as a historical and critical account but as a text book. The fundamental principles are not merely described as they were invented by the original investigators; they are rearranged, modified where necessary, and illustrated by experiments or examples worked out with a detail which the student could easily follow and imitate in other problems. For instance to enforce upon the mind the use of the Newtonian equation of motion

$$m\ddot{x} = F$$

the author solves in all detail — and remarkably neatly too — the problem : To determine the motion of two bodies connected by a spring which are constrained to move in a straight line, and of which one body is acted upon by a constant force. Again in the case of a body sliding down an inclined plane, which is itself free to move on rollers over a horizontal plane, the solution is accomplished in four different ways to illustrate the respective merits of four different methods. Even such questions as Lagrange's uniform method for the solution of any problem in statics or kinetics and Hamilton's principle of least action are not described without adequate explanation by examples.

Throughout the volume, where problems are solved or where they are not, there is a constant and successful attempt to render the subject tangible as well as knowable. The author states that until the solution has been seen and felt, the object of study has not been quite accomplished. A mere formal solution alone will not suffice.

In treating the foundations of dynamics Professor Mach gives Newton full credit and fair play. He not only quotes Newton's definitions, postulates, and laws; but adds many of his comments which usually are made with greater insight into the difficulties of the questions involved than most of the examples and explanations which subsequent authors have seen fit to substitute. Remarking on Newton's method of treatment he says: "We literally *see through* the cases of equilibrium and motion which occur. * * * The Newtonian conceptions are certainly the most *satisfactory* and the most lucid : and Poinsot shows a noble sense of scientific clearness and simplicity in making these conceptions the sole foundation of the science. * * * We join with the eminent physicists Thomson and Tait in our reverence and admiration of Newton." He is, however, not willing to stop here. He retracts a little with qualifications, adding: "We can only comprehend with difficulty their opinion that the Newtonian doctrines still remain the best and most philosophical foundations of the science that can be given." Then the author gives as follows his enunciation of the Newtonian principles.

"*a*. *Experimental Proposition:* Bodies set opposite each other induce in each other, under certain circumstances to be specified by experimental physics, contrary *accelerations* in the direction of their line of junction. (The principle of inertia is included in this.)

"*b. Definition:* The mass-ratio of any two bodies is the negative inverse ratio of the mutually induced accelerations of those bodies.

"*c. Experimental Proposition:* The mass-ratios of bodies are independent of the character of the physical states (of the bodies) that condition the mutual accelerations produced, be those states electrical, magnetic, or what not; and they remain, moreover the same whether they are mediately or immediately arrived at.

"*d. Experimental Proposition:* The accelerations which any number of bodies A, B, C, * * * induce in a body K, are independent of each other. (The principle of the parallelogram of forces follows immediately from this.)

"*e. Definition:* Moving force is the product of the mass-value of a body into the acceleration induced in that body."

These definitions have been quoted at length because in point of time and authority they have become historic and may in the future remain the most important changes suggested for bettering Newton's treatment. These definitions have been adopted more or less completely by such eminent authorities as Pearson in his Grammar of Science, Boltzmann in his Lectures on Mechanics, Volkmann,* and Slate.† On the other hand the definitions have been attacked severely by some authors, and these attacks have been answered in some of the numerous appendices to the volume, where Professor Mach endeavors to show that the attacking parties either furnish nothing better or agree with him more closely than they imagine.

It is not proposed to enter here upon the discussion, at any rate not from the standpoint usually followed. To do that would require a monograph. We merely desire to ask the questions whether these statements (a, b, c, d, e) of Professor Mach's, which more than anything else have been the cause of the violent discussions on the foundations of mechanics carried on during the few past decades, are in any essential way an improvement on Newton's. The difference between the two presentations is not that one has an experimental basis and the other not. Newton was a great experimenter. One difference is this: Newton uses, without exact definition, words which irresistibly suggest to us perfectly definite conceptions, the cor-

* Einführung in das Studium der theoretischen Physik, etc. Reviewed in the BULLETIN, October, 1902.

† Principles of Mechanics. Reviewed in the BULLETIN, May, 1902.

rectness of which is made apparent by comparing the results deduced theoretically with those observed by experience; Professor Mach and his followers invert the order, putting first the definite physical experiments phrased in such vague terms as *set opposite*, *induce* and *under circumstances to be specified*, wholly disregarding *à priori* considerations, assuming that inasmuch as a basis of experiment has been postulated the subsequent development of mechanics is assuredly accurate.

There is a noteworthy difference between French and German scientists on this point of mass and force. The philosophical and scientific speculations of the French favor for evident psychological reasons the introduction of force first and mass second. We may refer to Poincaré's La Science et l'Hypothese as giving the most recent and fundamental French views which will be found an instructive contrast to the German presentation of Mach.

There is one other point, near the end of the text, which merrits a bit of attention. On page 502 the author states that he will "attempt to show that the broad view expressed in the principle of the conservation of energy is not peculiar to mechanics but is a condition of sound scientific thought generally." The argument seems vague and feeble. It is too definite in some ways and insufficiently so in others. Happily we may avoid the detailed discussion by again referring to Professor Poincaré's work, in which we find among other things the statement: "If the universe is governed by laws expressible by mathematical formulas there must be something which is invariant." This is about as much and about as little as a conscientious scientist of to-day can say.

It would be wrong to infer from these adverse criticisms that the book is not to be most heartily recommended. In fact we believe we have shown the book in its most satisfactory aspect in order that afterward we might recommend it unqualifiedly and without deceiving any one. Owing to its clearness the work is easily read and might even be placed in the hands of students just beginning mechanics. In such a case certain sections might at first be omitted as too subtle or perhaps too uninteresting, but what remained would amply repay perusal by every student and teacher.

The French are to be congratulated that the work is soon to appear in their language. It is only to be hoped that the excellent typography, the full marginal references, the exhaustive index, and the moderate price, which all yield so much to the usefulness of the present edition, may also be a feature of the French edition. EDWIN BIDWELL WILSON.

ÉCOLE NORMALE SUPÉRIEURE,
PARIS, *December*, 1902.

FORSYTH'S DIFFERENTIAL EQUATIONS.

Theory of Differential Equations. By A. R. FORSYTH. Part II. *Ordinary Equations, not Linear*, Volumes 2 and 3; Part III. *Ordinary Linear Equations*, Volume 4.

IT becomes necessary from time to time to sum up in a work of considerable volume the knowledge which has accumulated in a certain field. The theory of differential equations is in some respects the most important part of mathematics. It is in this field that the astronomer and physicist most frequently appeals to the mathematician for assistance; for his problems, when finally formulated, usually assume the form that a certain differential equation is to be integrated. The most important transcendental functions, too, have been furnished to the mathematician by the integration of differential equations. No wonder then, that the literature is extensive, and there can be no doubt that mathematicians will feel grateful to Professor Forsyth for having lightened for them the labor of becoming acquainted with the labyrinth of investigations which have been carried on in this field.

That Professor Forsyth should have chosen to treat the linear equations last, may have been due to the fact that other works existed which treat of them in a modern and adequate manner. But systematically, and historically they should come first, as almost every question in regard to non-linear equations, that has been answered, has been suggested by the theory of linear equations. A student, therefore, would do well to read volume 4 of the present work first. Fortunately volume 4 has been so written as to enable him to do so. But we cannot pass this distinction between linear and non-linear equations by without remarking that there is no better way to convince ourselves of our ignorance on the subject of non-linear differential equations, than by studying Part II of Professor Forsyth's book. We find here, gathered with the greatest erudition, practically all