A MODERN FRENCH CALCULUS.

Cours d'Analyse Mathématique. Par ÉDOUARD GOURSAT, Professeur à la Faculté des Sciences de Paris. Tome I. Gauthier-Villars, Paris, 1902.

THE revision of the fundamental principles of the calculus, which was initiated by Cauchy and Abel and carried through by Weierstrass and his followers, led to the development of the ϵ-proof (early introduced by Cauchy) and to the precise formulation of definitions and theorems. In Germany and Italy a tendency sprang up to place only such restrictions on definitions and theorems as are necessarily imposed by the nature of the case. Thus functions continuous throughout no interval whatever were admitted as the integrand of a definite integral simply because the form of the definition of the integral applied to a certain class of these functions, and the question was examined of how far the ordinary theorems of the integral calculus hold for such functions. Again, the theorem that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ was proved with fewer restrictions than the continuity of all the derivatives that enter. While this procedure is perfectly justifiable so far as it is a question of research in a special field, it is important not to lose sight of the fact that investigations of this sort are but a very special phase of modern analysis, and that even the specialist in the field of analysis may never need to trouble himself about the integrals of other functions than those which are continuous except at a finite number of points. That which is essential for every mathematician to know who has occasion to use the calculus to any extent is a simple formulation of the theorems and simple tests for the validity of the processes of the calculus which have been handed down to us from Euler's time and earlier: — when may a convergent series of continuous functions be integrated term by term, when may a definite integral whose integrand satisfies reasonable conditions of continuity be differentiated under the sign of integration? These are questions of general interest to mathematicians. To the importance of a simple and lucid answer French mathematicians are alive. With full appreciation of modern standards of rigor they do not allow themselves to obscure in their presentation of the calculus the main facts of analysis by cumbersome details.

The book before us belongs to the best type of modern French treatises on the calculus. It is based on Professor Goursat's university lectures. According to the plan of instruction in France the student of mathematics learns at the *lycée* the meaning and use of derivatives, differentials not being introduced, and the rudiments of algebraic analysis.* Thus the university teacher can assume that the student is familiar with the notion of the limit and the elementary methods of the calculus, and that he has sufficient maturity to understand a treatment of the calculus such as is given in American universities in a second and third course.

After some introductory paragraphs, in which Rolle's theorem and the law of the mean are given, the author proceeds to a systematic treatment of partial differentiation, based on the total differential of a function. The fundamental theorems on which the properties of such differentials rest and which are overlooked in English text-books on the calculus † are here given their proper place.‡

Chapter II, pages 40–100, begins with the existence theorem for implicit functions, the proof being that which Dini gave in his lectures of 1877/78; and then follows the differentiation of the functions thus defined, properties of the Jacobian determinant, and change of variable. In algebra and algebraic analytic geometry the mere rudiments of partial differentiation suffice for the applications that arise; but in differential geometry and mathematical physics this is not the case. It is highly desirable that partial differentiation should be studied more at length than is at present the case, and a complete and lucid treatment such as is here given § will aid the teacher in modifying his course in that direction.

In American colleges students of calculus are not mature enough to appreciate existence theorems at the time when they

* *Cf.* Pierpont, "Mathematical instruction in France," BULLETIN (2), 6 (1900), p. 225.

† *Cf.* The writer's review: "A modern English calculus," BULLETIN (2), 8 (1902), p. 253.

‡ *Cf.* § 16, top of p. 29, and the corresponding theorem in § 14.

§ The present treatment is much fuller than that of Jordan, Cours d'analyse, and is illustrated by numerous applications of substantial character. A few examples taken from thermodynamics would have been a useful addition.

study partial differentiation and it is best to assume that equations defining implicit functions can be solved and lead to functions which have derivatives. It is easy, however, in using Professor Goursat's book, to assume these theorems at this stage and take them up later. In a logical development of the calculus they belong where the author has put them and for the working mathematician the arrangement here adopted is the most satisfactory one.

A few matters of detail before leaving these chapters. On page 5, line 12 from the top, the words "et à rester" should be inserted before "supérieure." The definition of approaching a limit in the case of a function of several variables is not given; the continuity of such a function is however carefully defined. The law of the mean for functions of several variables might well have been given a place in Chapter I. It appears nowhere explicitly, merely as a special case of Taylor's theorem with the remainder in Chapter III. On page 45, near the bottom, it is not of course true that if the three partial derivatives vanish simultaneously, the point is necessarily a singular point. This oversight occurs at various other places in the book. In the theorem at the beginning of § 28 it is necessary for the proof that for the system of values of $(u_1, \cdots, u_n)$ considered the partial derivatives $\Pi_{u_1}, \cdots, \Pi_{u_n}$ should not vanish simultaneously. This requirement should be made in the statement of the theorem. In this section (§ 28) the author does not bring out with all the clearness one could wish the fact that there are in all three cases: (*a*) the case in which the Jacobian determinant does not vanish; (*b*) the case in which it vanishes identically; and (*c*) the case in which it vanishes at a given point, but not at all points in the neighborhood. Cases (*a*) and (*b*) are treated. But little is known about case (*c*), and there is no reason for considering it; but the classification is important. A large number of examples are given at the end of Chapters I and II, all of which are taken from life. While the student will need to do many easy examples, like those given in English books on calculus, at the start, he will have the satisfaction in working these more difficult ones of knowing that they are not artificial, that they truly illustrate the actual applications of partial differentiation.

Chapter III is devoted to Taylor's theorem with the remainder and to Taylor's series, both for functions of one and for functions of several variables, and to applications. The

French have not committed themselves to the exclusive use of power series.* When it is simpler to use Taylor's theorem with the remainder, they do so; and it is an excellent feature of the book before us that the infinite series is not employed when the finite series is better suited to the purpose. Indeterminate forms are treated by the use of Taylor's theorem, but we miss the more general theorem that if $f(a) = \phi(a) = 0$ or ∞, and if $f'(x)/\phi'(x)$ approaches a limit, then $f(x)/\phi(x)$ approaches a limit and $\lim f(x)/\phi(x) = \lim f'(x)/\phi'(x)$. Thus such limits as

$$\lim_{x=0} x \log x \qquad \text{or} \qquad \lim_{x=\infty} \frac{\log x}{x},$$

or more generally

$$\lim x^{\alpha} (\log x)^{\beta},$$

when $x = 0$ or ∞; or again

$$\lim_{x=\infty} x^{n} e^{-x}$$

must each be evaluated by a special investigation,—one for which no general method is given.

The treatment of maxima and minima of functions of several variables is admirable, adequate and clear, but not overdone.† The examples studied are well chosen and include clothed problems that do not come under the ordinary rule.

Chapters IV–VII, pages 150–368, are entitled respectively Definite Integrals, Indefinite Integrals, Double Integrals, and Multiple Integrals, Integration of Total Differentials. In American colleges the custom is growing of introducing the definite integral, defined as the limit of a sum and represented by the area under a curve, early in the first course in calculus, and of applying it to the solution of a variety of problems in physics and mechanics. Thus the student comes to the second course in calculus with a pretty good idea of what integrals are

* This tendency in Germany is well hit off by Bohlmann in his report on important works on the calculus, *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 6 (1897), p. 110. One of the entries in the table of contents reads "Es gibt nur Potenzreihen!"

† The definition of a maximum or a minimum on page 130 would seem to exclude the case that $f(c+h) - f(c)$ can vanish for other values of h than the value $h = 0$; but this is not the definition employed on page 136: "Le cas où $b^2 - ac = 0 \cdots$."

and what they are for, and he is prepared for the study of double integrals, improper integrals and functions defined by integrals. It is at this stage, that is, in the second calculus course in American colleges, that these chapters of Professor Goursat's book may be employed with great advantage. They give a simple and rigorous elementary treatment of the subjects just mentioned — particularly the chapter on double integrals, pages 282–333 — a masterpiece of presentation, so clear and rigorous and to the point that it may well serve as a text in the treatment of this subject. Regarding the first of these chapters a similar remark applies to the one made concerning the existence theorem for implicit functions in Chapter II; namely, that in the second course in calculus it is well to assume the theorems about continuity, the proof coming more properly at a later stage, when questions of uniform convergence are treated; but here again the arrangement which the author has adopted will commend itself to the working mathematician as being the natural sequence of topics. There is one omission that seems to us unfortunate. It is that of Duhamel's theorem regarding the representation of any infinitesimal in a limit of a sum by another infinitesimal that differs from it by one of higher order. But as we are treating this subject at length elsewhere, we restrict ourselves here to a reference.*

The following topics considered in these chapters deserve special mention because of the lucid and rigorous treatment: change of variables in simple and multiple integrals §§ 84, 128,† 130, 145, 150; improper integrals simple and multiple,

* *Annals of Math.* (2), 4 (July, 1903).

† The method here set forth of proving by means of Green's theorem that when curvilinear coördinates $x = f(u, v)$, $y = \phi(u, v)$ are introduced, the area Ω bounded by the curves $u = u_0$, $u = u_0 + du$, $v = v_0$, $v = v_0 + dv$ is

$$\Omega = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} du\, dv,$$

where the Jacobian is to be formed for a certain point (u', v') in Ω, is especially elegant. The author's reason for not extending it to triple integrals was doubtless that the formulas become a little long; but they present no serious difficulty and the generalization may well serve as an exercise for the student, his attention being first called to the fact that the three-rowed Jacobian may be written in the form

§ 89*–91, 133; line and surface integrals, including Green's and Stokes's theorems, §§ 93, 126, 135, 136, 149, 151–155: differentiation under the sign of integration, § 97; approximate calculation of an integral by Simpson's and Gauss's rules; Amslers's planimeter, § 99–102; the geometric interpretation of the method of rationalization employed for computing the integrals $\int R(x, \sqrt{a + bx + cx^2})dx$, § 105.

Chapters VIII, IX, pages 369–478, deal with infinite series. The subject is treated *ab initio* and the ordinary tests for convergence and theorems relating to the algebraic transformation of series are developed with the clearness and rigor that mark the whole book. But the last page of § 169 relating to double series whose terms are not all of like sign needs to be expanded. — The definition of uniform convergence in the case of infinite series is the one given by Darboux,† and is as follows: The series of continuous functions

$$u_0(x) + u_1(x) + \cdots,$$

convergent in the interval (a, b), is said to be uniformly convergent in this interval if to every positive ϵ there corresponds a positive integer n, independent of x, such that the remainder

$$R_n(x) = u_n(x) + u_{n+1}(x) + \cdots,$$

remains in absolute value less than ϵ for all values of x in the

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$$\begin{vmatrix} f_u & f_v & f_w \\ \phi_u & \phi_v & \phi_w \\ \psi_u & \psi_v & \psi_w \end{vmatrix} = \frac{\partial}{\partial u}\begin{vmatrix} f_v & f_w \\ \phi_v & \phi_w \end{vmatrix} + \frac{\partial}{\partial v}\begin{vmatrix} f_w & f_u \\ \phi_w & \phi_u \end{vmatrix} + \frac{\partial}{\partial w}\begin{vmatrix} f_u & f_v \\ \phi_u & \phi_v \end{vmatrix}.$$

The possibility of this extension was pointed out by the author in his original paper, *Bulletin des sciences math.* (2), 18 (1894), p. 72.

* The theoretical developments here given are especially simple. The rule for the convergence of the integral

$$\int_a^b f(x)\,dx,$$

when $f(x)$ is discontinuous at the point $x = a$, page 198, bottom, can, however, be formulated still more simply if we require merely that for some value of $0 < k < 1$ the variable $(x-a)^k f(x)$ shall approach a limit when x approaches a. A similar remark applies to the formulation of the other tests for convergence and divergence of improper integrals.

† *Ann. École norm.* (2), 4 (1875), page 77.

interval. It will be observed that this definition differs from the one ordinarily given in that it does not require that $|R_n(x)| < \epsilon$ for *every* $n > m$, but only for *one* value of n. Nevertheless the proof that such a series represents a continuous function is sound. The proof that the series may be integrated term by term is, however, invalid, as the following example shows: let

$$S_n(x) = nxe^{-nx^2} \text{ when } n \text{ is odd};$$
$$S_n(x) = 0 \quad \text{`` } n \text{ `` even.}$$

Here the term by term integral in the interval (0, 1) has the value

$$\int_0^1 S_n(x)dx = \tfrac{1}{2}(1 - e^{-n}) \text{ when } n \text{ is odd};$$
$$= 0 \quad \text{`` } n \text{ `` even.}$$

What is true is this: if such a series be integrated term by term and parentheses then be suitably introduced, the series of parentheses will converge toward the integral of the function as its limit. Tannery * has called attention to the fact here mentioned, namely, that a series uniformly convergent according to Darboux's definition can not always be integrated term by term, but both Picard † and Goursat appear to have overlooked Tannery's corrections. In § 175, however, which deals with the differentiation of an improper integral

$$\int_a^\infty f(x, \alpha)dx$$

under the sign of integration, the definition of the uniform convergence of such an integral is the one ordinarily given. If such an integral converges uniformly (and the integrand is continuous in x, α), it represents a continuous function of α and may be integrated under the sign of integration:

$$\int_a^{\alpha_1} d\alpha \int_a^\infty f(x, \alpha)dx = \int_a^\infty dx \int_{\alpha_0}^{\alpha_1} f(x, \alpha)d\alpha.$$

These theorems are precisely analogous to the theorems about

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* Fonctions d'une variable, Paris, 1886, p. 366, foot-note. The foot-note of pages 133–4, so far as it relates to integration, contains an inaccuracy, which is corrected in the later foot-note just cited.

† Traité d'analyse, vol. I, chap. VIII, 1st ed., p. 195; 2d ed., p. 211.

uniformly convergent series and should be given along with the sufficient condition the author deduces for differentiating under the sign of integration. The theorems relating to the transformation of power series in one and in several variables, and the theorems that follow from these, together with the proof of the analytic character of the implicit functions defined by a system of analytic equations, $F_i(x_1, \cdots, x_q; y_1, \cdots, y_p) = 0$ $(i = 1, \cdots, p)$ are all faithfully developed, pages 419–461 being devoted to this topic. The method of the *fonctions majorantes* is explicitly set forth. Analytic functions of real arguments are defined by the property that they admit a development by Taylor's theorem. The author does not neglect to remind the reader that in spite of the important rôle that these functions play, they form after all but a very special group of real functions of real variables within the general group of continuous functions. Finally the development of a continuous function into a Fourier's series is established. The volume ends with three chapters, pages 479–610, on applications of the differential calculus to curves and surfaces.

The extraordinarily high standard of simplicity and attractiveness in style, combined with modern rigor, which Picard set in his Traité d'analyse is fully maintained by Professor Goursat. The objects of the two works are quite different. Picard's purpose was to write a treatise on differential equations, and he developed only such parts of analysis and geometry as bear on this subject. Goursat, on the other hand, has set himself the task of writing a systematic treatment of the calculus, and thus the whole field of the calculus is included here.

A treatise on advanced calculus which should present the whole subject rigorously and attractively, and at the same time in the spirit of modern analysis, has been sorely needed by students of mathematics who intend to proceed to the study of mathematical physics or of some of the various branches of analysis — theory of functions, differential equations, calculus of variations, etc. Professor Goursat's work meets the needs of such students in a thoroughly satisfactory manner, and we recommend it to them most heartily. The teacher of calculus will find many suggestions in the book which will enable him to improve his course, and he may often with advantage refer even an elementary class to the more elementary parts of the book for collateral reading. The range of the book is wide.

While beginning with the elements of the calculus, it carries the reader to the point where he is prepared to use original sources and extracts from ϵ-proofs the underlying thought. When the future historian inquires how the calculus appeared to the mathematicians of the close of the nineteenth century, he may safely take Professor Goursat's book as an exponent of that which is central in the calculus conceptions and methods of this age. W. F. OSGOOD.

HARVARD UNIVERSITY,
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SHORTER NOTICES.

Niedere Zahlentheorie. Erster Teil. By Dr. PAUL BACHMANN. B. G. Teubner's Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Band X, 1. Leipzig, 1902. x + 402 pp.

IN view of the ambitious series of volumes by Bachmann, giving a comprehensive exposition of number theory, a series not yet completed, the appearance of a new volume on the elements of the subject, quite independent of the series mentioned, will doubtless cause some surprise. When the invitation came to the author to contribute to Teubner's Sammlung a text upon the subject on which he is so eminent an authority, he hesitated long, fearing that a text on the elements of number theory ran the risk of conflicting with his Elementen. The author has attempted to avoid this conflict in two directions: first by the addition of much important material; second, by employing a method of construction different at least in essential points. The author believes that the present book, both in contents and in foundation, may well be considered as a supplementary volume to his former series. As indicating in detail parts differing essentially from the Elementen, there may be mentioned the chapter on the different euclidean algorithms, including Farey's series, the theory of binomial and general congruences, the exhaustive treatment of the known proofs by elementary number theory of the quadratic reciprocity law and the interrelations of these proofs. The theory of higher congruences is appropriately introduced, even in the Niedere Zahlentheorie, both by way of climax to the elementary parts and to afford a satisfactory insight into the means employed by Gauss in his seventh proof of the reciprocity law.