THE CHARACTERIZATION OF COLLINEATIONS.

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A COLLINEATION is ordinarily defined as a point transformation which converts collinear points into collinear points, *i. e.*, one for which the family of ∞^2 straight lines, or the equivalent differential equation $y'' = 0$, is invariant. The question suggests itself whether this definition does not contain redundancies—whether it is not sufficient to require that only some straight lines shall remain straight. If we understand by a *simple* system of lines any system possessing the property that through each point of the region of the plane considered there passes one and only one line, then the result of this note may be stated:

If four simple systems of straight lines remain straight after a point transformation, then the same is necessarily true of all straight lines, and the transformation is, therefore, a collineation.

To prove this consider the general point transformation T

$$X = \phi(x, y), \quad Y = \psi(x, y), \tag{1}$$

where ϕ and ψ are one-valued continuous functions possessing first and second derivatives in the region considered. It is assumed, of course, that the Jacobian, $J = \phi_x\psi_y - \psi_x\phi_y$, does not vanish identically. The once extended transformation is

$$Y' = \frac{\psi_x + \psi_y y'}{\phi_x + \phi_y y'}; \tag{2}$$

and the twice extended may be written *

$$Y'' = \frac{\alpha + \beta y' + \gamma y'^2 + \delta y'^3 + Jy''}{(\phi_x + \phi_y y'^3)}, \tag{3}$$

where

$$\begin{aligned} \alpha &= \phi_x\psi_{xx} - \psi_x\phi_{xx}, \\ \beta &= \phi_y\psi_{xx} - \psi_y\phi_{xx} + 2\phi_x\psi_{xy} - 2\psi_x\phi_{xy}, \\ \gamma &= \phi_x\psi_{yy} - \psi_x\phi_{yy} + 2\phi_y\psi_{xy} - 2\psi_y\phi_{xy}, \\ \delta &= \phi_y\psi_{yy} - \psi_y\phi_{yy}. \end{aligned} \tag{4}$$

* Lie-Scheffers, Continuirliche Gruppen, 1893, p. 33.

If now T transforms any straight line into a straight line, then for that line $y'' = 0$ and $Y'' = 0$, so that from (3)

$$(5) \qquad \alpha + \beta y' + \gamma y'^2 + \delta y'^3 = 0.$$

This differential equation of the first order and third degree is satisfied by all the invariant straight lines. Assuming that there are four simple systems of such lines, then through each point there pass four; these give four values of y', corresponding to fixed values of $\alpha, \beta, \gamma, \delta$. Equation (5) is therefore an identity. This means, however, that $Y'' = 0$ is a consequence of $y'' = 0$, or that $y'' = 0$ is invariant under T.* This proves the result stated above.

The result may be restated in terms of differential equations as follows. A simple infinity of straight lines is represented by a Clairaut differential equation

$$(6) \qquad y = xy' + f(y'),$$

where y' is a one-valued function of x, y. If an equation of this kind is transformed into a similar equation, then the corresponding lines remain straight. Hence, *if T transforms four Clairaut equations of the first degree into Clairaut equations, then the same is true of all Clairaut equations, and T is a collineation.*

It may be added that what has been obtained is, in a sense, *the minimum definitive property of the collineations.* For there exist non-projective transformations which possess one, two, or three simple systems of invariant straight lines. A familiar example of the latter type is given by inversion with respect to a circle; here the three invariant systems are the pencils through the circular points at infinity and through the center of the circle. An example where all three systems are real is given by the quadratic transformation $X = 1/x$, $Y = 1/y$, which leaves invariant the systems $x =$ const., $y =$ const., $y/x =$ const. The same systems are invariant under the three parameter group $X = ax^c$, $Y = bx^c$.

The problem of determining all transformations possessing three invariant systems of straight lines is one of considerable difficulty. The writer expects to discuss it, as well as the extension of the results to space, in a future paper.

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* The direct integration of the equations $\alpha = \beta = \gamma = \delta = 0$, leading to collineations, was accomplished by Scheffers; cf. p. 34, l. c. For infinitesimal transformations the analysis is simpler. Lie-Scheffers, Differentialgleichungen, 1891, p. 389.