THE THEORY OF AUTOMORPHIC FUNCTIONS.

Vorlesungen über die Theorie der Automorphen Functionen. Von ROBERT FRICKE und FELIX KLEIN. Leipzig, B. G. Teubner. Bd. I: *Die gruppentheoretischen Grundlagen,* 1897, xiv + 634 pp. Bd. II: *Die functionentheoretischen Ausführungen und die Anwendungen;* Erste Lieferung: *Engere Theorie der Automorphen Functionen,* 1901, 282 pp.

SIDE by side with the growth of general function theory special classes of functions have developed in which the general theories have found abundant opportunity to display their fertility and power. On the other hand, the study of special functions has repeatedly afforded the stimulus and suggested the path for new investigations along general lines. When, in addition, the intrinsic value and usefulness of such functions as the elliptic, hyperelliptic, abelian, hypergeometric, Bessel, etc., is taken into consideration, it is readily seen that the cultivation of such special fields is scarcely second in interest and importance to that of the general theory itself.

Among the various classes of special functions which have hitherto engaged the attention of mathematicians, that of most recent origin, and of by far the largest content (at least potentially) is the automorphic functions. This vast subject, the growth of the past quarter of a century, owes its rapid development to the genius and assiduity of the two eminent mathematicians, Klein and Poincaré, as well as to the comparatively high state of perfection of other mathematical disciplines which have been forced to contribute their assistance to this new field. Klein and Poincaré have each brought to this subject a breadth of knowledge and a corresponding wealth of ideas truly remarkable at the present day when multiplicity of interests hardly permits the investigator any other choice than to specialize within limits more or less narrow.

Klein in particular has by precept and example urged the importance of a closer unity among all departments of mathematical thought, and the great advantage to be derived from bringing to the aid of any one field the combined resources of all the others. It is in his work on the automorphic functions that he has given the most brilliant illustration of this mode of treatment. In this work he has been ably seconded by the efforts of the pupils and co-laborers he has been so fortunate as to gather about him at the Göttingen school, among whom Robert Fricke is especially deserving of the highest praise for the ability and industry which he has brought to the arduous labor of preparing for publication the volumes now before us, and for the wealth of material which he himself has contributed to the subject.

These volumes form the sequel to, and final elaboration of the ideas contained in the volumes previously published, namely, the Ikosaeder (by Klein), and the Elliptische Modulfunctionen * (by Klein and Fricke). The former of these works deals exhaustively with the finite groups of transformations on a single variable, and the functions associated with them. The two large volumes of the latter treat in a very elaborate manner the modular group, and the general ideas necessary to be followed out in the study of all such groups and their functions. One of the principal objects of this fullness of discussion in the Modulfunctionen is to pave the way for a subsequent discussion of the general theory of automorphic functions without the restraint of burdensome details.

The advantage of thus disposing of preliminary details, and familiarizing the reader with many of the fundamental ideas as applied to a concrete example is clearly perceived when we observe how completely are the various elements in the present work fused into an organic whole, each part in strong and vital contact with every other part. The degree of abstractness and concentration thus attained is necessary for a forceful treatment of a subject which seeks to embrace in one view such an infinite variety and complexity of phenomena.

The theory of automorphic functions has for its object to investigate the discontinuous groups of linear transformations of a single variable ζ, and to study the properties of functions which are invariant for the transformations of any given group.

The most important aid in this work comes from geometry, † which affords a concrete representation of the values of the

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* The interesting review of the Modulfunctionen by Professor Cole, and the general survery of Klein's work and ideas which it contains, renders it unnecessary for us to give more than a passing notice to the antecedents of the present work. See BULLETIN, 1st Series, vol. 1 (1892), p. 105.

† Geometry is, in fact, at the present stage of development of the subject, an indispensable tool. To the lack of geometrical aids is in no small measure to be attributed the meager success that has attended the various attempts to investigate automorphic functions of more than one variable.

complex variable ζ and pictures the effect of a group of transformations applied to this variable. In the Automorphe Functionen an introductory chapter of about sixty pages is accordingly devoted to the discussion of some geometrical notions not explained in the Modulfunctionen.

The first geometrical interpretation of the complex variable ζ is serviceable only for a restricted, but very extensive class of groups, the *fundamental circle* groups (Hauptkreisgruppen). The values of ζ are associated with the homogeneous coördinates (z_1, z_2, z_3) of points in a plane by means of the relation

$$= \zeta \frac{z_2 + \sqrt{z_2^2 - z_1 z_3}}{z_3}.$$

Collineations which leave the conic

$$(1) \qquad z_2^2 - z_1 z_3 = 0$$

unaltered correspond to linear transformations

$$(2) \qquad \zeta' = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta}$$

on ζ. Taking the conic (1) as the absolute for a system of non-euclidean measurement of distance, the transformations (2) leave distances unaltered, and hence they are called (in a generalized sense) motions. Congruent figures, that is, figures which transform into each other have like (non-euclidean) areas. The interior of the conic, regarded for convenience as an ellipse, is called the "hyperbolic" plane.*

Out of the ∞^6 transformations of the continuous group (2) let us suppose a discontinuous group Γ selected by any suitable definition. This group is then *properly*, or *improperly* discontinuous, according as the fundamental region for Γ has a finite or an infinitesimal area. By "fundamental region" is meant a division of the hyperbolic plane such that no two points of the region are congruent, while every point without the region is congruent to some point within. The properties

* The cases in which the conic is imaginary (giving the "elliptic" plane), or breaks up into the circular points at infinity (giving the ordinary, or "parabolic" plane) lead to groups of finite order, or groups associated with the elliptic functions. These cases receive their full share of attention in the work under review, but from lack of space we pass them by without discussion.

of a given group are all implicitly involved in the geometrical properties of the fundamental region which it defines, and, accordingly, a large portion of what follows in volume I is devoted to a geometric study of the fundamental region and the network of congruent regions which arise from it by transformations of the group.

Although, evidently, the fundamental region can be selected with a high degree of arbitrariness, it can always be defined (and this in an infinity of ways) so as to be bounded by straight lines. The fundamental region is then a polygon, and the group belonging to it is called a polygon group.

A particularly useful choice for a fundamental polygon, and one which plays a leading rôle throughout the book, is determined as follows: Let C_0, C_1, C_2, $\cdots$ be any set of congruent points. Around the points C_i construct circles K_i of equal radii r, meaning by circle in this connection the locus of all points whose non-euclidean distance from a given C_i is constant. Assuming r at first sufficiently small so that no two circles collide, let the radii increase simultaneously and at the same rate for all the circles K_i with the understanding that the lengthening of any particular ray emanating from C_i shall cease the instant it meets a like ray emanating from a neighboring point C_k. Imagine this process to continue until the entire hyperbolic plane is filled without gap and without overlapping. The locus of the end points of the rays departing from a center C_i will consist of a closed chain of straight lines forming the boundary of a convex polygon. This region, called the *normal polygon*, is evidently a fundamental region for the group.*

Now let the values of ζ be associated in the ordinary way with the points of a complex plane. Then, assuming $z_2^2 - z_1z_3$ to be negative for the interior of the ellipse, to each point of the hyperbolic plane correspond two points of the ζ-plane representing conjugate imaginary values of ζ. To each point on the ellipse corresponds one point on the real ζ-axis, and to

* Each of the points C_i is the center of a normal polygon. This polygon clearly contains within itself all points which are nearer to C_i (in the non-euclidean sense) than to any congruent point. Each polygon is equal in all respects to every congruent polygon (still speaking in the non-euclidean sense). This picturing of the group with a network of equal polygons filling up the hyperbolic plane, is seen to be a generalization of the idea of a network of equal parallelograms filling the ordinary (or parabolic) plane, with which the theory of elliptic functions has made us familiar.

each point outside the ellipse corresponds a pair of points on the real axis. On account of the one-to-two correspondence it is evidently sufficient to take into consideration only half of the ζ-plane. Moreover, by a linear transformation of ζ the real axis changes into a circle C the interior of which has a one-to-one correspondence to the half ζ-plane and hence to the hyperbolic plane.

In the hyperbolic plane the polygons crowd together in infinite number along the ellipse which forms its boundary, since this is the infinite element in the plane. Any set of congruent points, such as the C_i above mentioned, will have limiting points, or points of accumulation, on this boundary. These limit points may cover the ellipse everywhere densely, in which case the ellipse, and correspondingly the circumference of the circle C, forms a natural boundary for the group and its network of polygons.* The group is then called a *limit circle group* (Grenzkreisgruppe). On the other hand, segments of the ellipse (or circle C) of finite length may be free from limiting points. These two classes, which include all groups whose fundamental regions fill the hyperbolic plane, are specified by the general term, *fundamental circle groups.*

The foregoing method of associating the ζ values with the real points of the hyperbolic plane is of practical value only when the substitutions of the group have real coefficients since it is only in that case that real points in the hyperbolic plane are transformed into real points.* This difficulty is obviated by a more general procedure. A quadric surface is taken in ordinary space, for convenience the sphere

$$(3) \qquad z_1^2 + z_2^2 + z_3^2 - z_4^2 = 0.$$

The values of ζ are associated with points on the surface of the sphere in the familiar manner of the Riemann function theory by means of the relation

* It follows that any function, invariant for transformations of the group, has the circle for natural boundary and cannot be continued over this boundary which is everywhere filled with essential singularities of the function.

* That there is a practical necessity for the restriction to real coefficients is not mentioned by our authors, although they may have expected it to be inferred. This inference, however, is not likely to be at once drawn by the reader, since the reason assigned in the text is that only in this case are groups of "motion," or motion combined with inversion, obtained, without explaining why it is desired to exclude from consideration all groups other than these.

$$(4) \qquad \zeta = \frac{z_1 + iz_2}{z_4 - z_3}.$$

Every collineation which transforms the sphere into itself subjects ζ to a linear transformation of one of two kinds,

$$(5) \qquad \zeta' = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta}, \qquad \zeta' = \frac{\alpha\bar{\zeta} + \beta}{\gamma\bar{\zeta} + \delta},$$

when $\bar{\zeta}$ is the conjugate of ζ. Taking the surface of the sphere as the absolute for a system of non-euclidean measurement in space, the interior of the sphere is called the hyperbolic space. The non-euclidean distance between two points is unchanged by the transformations (5). Those of the first type are "movements" of the hyperbolic space, while one of the second type is a reflection, or symmetrical transformation with respect to a certain plane of symmetry.

The values of $\zeta = \xi + i\eta$ are likewise represented by the coördinates of points in a complex plane referred to ξ, η rectangular axes. A third rectangular coördinate ϑ in space is introduced. Then assuming

$$\vartheta = \pm \frac{\sqrt{z_4^2 - z_1^2 - z_2^2 - z_3^2}}{z_4 - z_3},$$

this equation combined with (4) establishes a correspondence between the ξ, η, ϑ space and the hyperbolic space such that to a point within the sphere correspond two points situated symmetrically with respect to the ζ plane. The portion of the ξ, η, ϑ space situated on one side of the ζ plane has a one-to-one correspondence with the hyperbolic space. This is called the ζ half-space. To each plane of the hyperbolic space corresponds a half sphere of the ζ half-space which is orthogonal to the ζ plane.

Consider now a discontinuous group Γ of transformations (5). A fundamental region for Γ will consist of a limited portion of the hyperbolic space (or of the ζ half-space) of largest possible extent subject to the condition that no two congruent points are within this region. Such a fundamental region can be selected in an infinity of ways so as to be bounded by planes alone. If this fundamental region be transformed by all the substitutions of the group, the totality of congruent regions thus obtained will fill the hyperbolic space without gap and without overlapping. These polyhedra may or may not intersect the surface of the sphere. In the former case the surface is divided up into a network of polygons forming fundamental regions for the ζ transformations, which, by the mediation of the above correspondence (4), may be depicted on the ζ plane, while at the same time the network of polyhedra are represented by polyhedra in the ζ half-space whose faces are half spheres orthogonal to the ζ plane. The group is then properly discontinuous in the ζ plane (or on the ζ sphere).

If, in particular, the group consists of all the transformations which leave unchanged a point P outside the sphere, the polar plane of P and its circle of intersection with respect to the sphere is invariant for the group, which is accordingly a fundamental circle group. From this point of view it is called a *hyperbolic rotation group*. If the point P is on the sphere, we have the *parabolic rotation groups* with one limit point, to which the elliptic functions are related. Finally, when P is within the sphere, the polar plane does not intersect the surface in real points, and hence the group has no limit points. There are accordingly only a finite number of fundamental regions, and the group is finite. It is called an *elliptic rotation group*.

It is found that the use of the polyhedra in the hyperbolic space is an especial convenience and simplification in the study of properties of non-rotation groups, and indispensable for the investigation of groups which are improperly discontinuous on the surface of the sphere (or in the ζ plane). An example of the latter kind to which considerable space is allotted is the Picard group which consists of all the substitutions of ζ formed with complex integer coefficients whose determinant is 1 or i.

Returning to the polyhedra into which the hyperbolic space has been divided by the transformations of the group, it is further noticed that these can be determined in a manner exactly analogous to the determination of the normal polygons in the hyperbolic plane. A set of congruent points C_0, C_1, $\cdots$ is selected at random and small spheres (in the non-euclidean sense) of equal radii r described about each. Then r is imagined to increase simultaneously for all these spheres, with the understanding that any ray emanating from C_i stops increasing when it meets a ray proceeding from a neighboring center. When this process is carried to an end, the hyperbolic space will be filled without gaps and without overlapping by polyhedra, all of whose faces are planes. A polyhedron determined in this way is called a *normal polyhedron*. The study of the normal fundamental polyhedra here plays the same rôle in investigating the properties of the group as the normal polygon does in the case of the fundamental circle groups.

Particular attention is devoted to those non-rotation groups whose normal polyhedra intersect the surface of the sphere in one or more distinct networks of polygons. A classification of groups is based on the nature of these nets and their limit points.

There may be a single net completely covering the sphere. The limit points are then isolated, and if there are more than two of them, the number is infinite. On the other hand, the number of nets may be two, or infinite. In either case the number of limit points is infinite since the nets are separated from each other by these points. Hence non-rotation groups may be classified into:* (*a*) those with two limit points; (*b*) those with an infinity of limit points. Division *b* is classified still further according as the number of nets is (1) *one*, the limit points being isolated; (2) *two*, the limit points forming a non-analytic curve separating the two nets; (3) *infinite*, the infinity of nets being separated from each other by an infinity of limit curves formed by limit points of the group. This last case is further subdivided according as the limit curves are (at least in part †) non-analytic, or consist entirely of circles, and still further as the nets are simply or multiply (always with an infinite multiplicity) connected.

A kind of converse question naturally arises, namely, in how far can an arbitrarily chosen polygon serve as the fundamental region for a group of linear transformations, and thus serve to define and generate the group? The restrictions under which a polygon can be chosen are determined for groups in the hyperbolic plane (fundamental circle groups). The like problem is then solved for polyhedra in the hyperbolic space and the results made useful for the determination of polygons suitable for defining non-rotation groups by means of the theorem: any polygon P_0 (on the surface of the sphere) is a fundamental

*These are divisions II and IV in the book. See pp. 164–5. Division I comprises the ordinary cyclic groups formed by the repetition of a single operation, and division III comprises the rotation groups already noticed above.

† If analytic curves occur, they must be circles.

region for some discontinuous group of transformations provided that a polyhedron can be constructed intersecting the sphere in P_0 and satisfying the restrictions which polyhedra are subject to in order to be usable for fundamental regions in the hyperbolic space.

A new kind of fundamental region, the *canonical polygon*, is next introduced which greatly facilitates the discussion of the generating operations of the group and the relations existing among them. To make clear the idea, it is observed that the sides of a fundamental polygon must be congruent in pairs, and the vertices in cycles. If the sides of a fundamental polygon be deformed so as to bring together and unite congruent lines and points, a closed surface F is obtained of a certain genus p which is called the genus of the group. The surface F may be conveniently thought of as an ordinary Riemann surface.

Consider now the reverse process. Take any Riemann surface and draw from an arbitrary point E a canonical system of cross-cuts which reduces the surface to a simply connected one. It may then be deformed into a polygon suitable for the fundamental region of a group. The opposite edges of the cross-cuts become congruent edges of the polygon, and the substitution which brings an edge into coincidence with a congruent edge is one of the generators of the group. If the fundamental polygon is assumed to have vertices which are fixed points for parabolic or elliptic substitutions, then a certain number of points $e_1, e_2, \cdots, e_n$ are selected in the Riemann surface and cuts drawn from E to each of these points. If e_i is to be a fixed point of an elliptic substitution of finite period, then the deformation must be carried out so that in the end the angle at e_i is an aliquot part of π. If, on the other hand, e_i is to be a fixed point for a parabolic substitution, the angle must be zero after the deformation. The polygon so obtained is the canonical polygon. It has $2n + 8p$ edges which are congruent in pairs. It is possible to unite some of the congruent edges by a suitable deformation of the polygon. The $n + 4p$ substitutions which transform congruent edges into each other are generators of the group. Certain relations exist among these which readily permit their number to be reduced to $n + 2p$ among which a single relation exists (in addition to the relations of the form $V^l = 1$ for the elliptic substitutions).

The second division of volume I is devoted to an application of the geometrical foundation principles, as laid down in the first division, to the detailed study of polygon groups. The normal and canonical polygons are made the leading instruments of investigation, and the cases of the elliptic and parabolic rotation groups, the non-rotation groups with two limit points, the hyperbolic rotation groups, and the non-rotation groups with an infinity of limit points are considered in turn. The first three cases are comparatively simple and easily disposed of. The fourth case is treated at great length. The important principle is first deduced that for any group Γ the normal polygon with a given center C is the region common to the normal polygons (with the same center) of all the groups whose composition generates Γ. This principle is applied in particular to the cyclic subgroups of Γ.

Various properties of the sides and corners are next considered. It is shown for example that, while at an accidental corner (that is, one which is not a fixed point for a substitution of the group) in general only three polygons meet, for special positions of the center C* more than three polygons meet in such a point. All polygons having the same genus p and number n of fixed vertices are said to be of the same kind and of character (p, n). Polygons of the same kind having a like number of sides, the sides being congruent in the same order, are said to be of the same *type*. The type is *ordinary* or *special* according as the accidental corners occur in cycles of three and the fixed corners in cycles of one, or not. The relations which are found to exist between corners, sides, and number of cycles show that the occurrence of a special type involves a reduction of the number of sides s. Special types with $s - 4$ sides occur with each group and arise from a particular choice of the center C. Special types in which the number of sides is diminished by six or more below the number belonging to the ordinary type can occur only for special groups. There are called *singular groups*.†

* In fact C will lie either on the ellipse, or on a certain curve of the third degree.

† Our authors do not attempt to develop a theory of singular groups, but content themselves with remarking on the possible importance of these groups in relation to the theory of algebraic functions. The reader of the Automorphe Functionen will have frequent occasion, as here, to notice the occurrence of gaps in the development of the subject. While the work under review may rightly be regarded as presenting a systematic and well-developed theory of the automorphic functions, a vast amount of work yet remains to be done. To mention only one further instance, it may be remarked that the theory of non-rotation groups has not yet been systematized, and in spite

The second chapter in this division of the book deals with the canonical polygon with special reference to the form it may take in the hyperbolic plane, and the transformations it may undergo on account of transformation from one canonical system of cross-cuts in the Riemann surface to any other. The details are first worked out for polygons of character (0, 3), (1, 1), and (0, n). The methods being made clear by these simplest cases, the case (p, n) is then treated in its generality. The remainder of the chapter is devoted to an extended discussion of the *moduli* (Moduln). The moduli are the parameters on which the substitution coefficients depend, and which are invariant when the group is transformed by any linear substitution. The groups so obtained are said to belong to the same class, and the moduli form a system of numbers characteristic for that class. As the groups of the same class have like structure, a determination of the properties of the class acquaints us with the properties of any group of the class. Any particular set of possible values for the moduli determines a class, and the number manifoldness of all the classes is measured by that of the moduli.

The third and last division of volume I treats of the arithmetic definition of discontinuous groups. This subject, although of fundamental importance, offers at the present time only fragmentary results on account of the great difficulties that it has to overcome.

The first chapter determines the arithmetic characteristics of the rotation subgroups of the Picard group. This is accomplished by the introduction of the Dirichlet and the Hermite quadratic forms, concerning which the following two theorems are proved: Every hyperbolic or loxodromic cyclic subgroup transforms into itself a particular Dirichlet form having for determinant a non-square; and, conversely, every such form is invariant for some such subgroup. Again, every fundamental circle group contained within the Picard group transforms into itself a particular indefinite Hermite form; and conversely, every such form is reproduced by some fundamental circle subgroup of the Picard group.

The second chapter discusses the groups that reproduce cer-

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of the considerable space devoted to this subject and the powerful aid afforded by the introduction of the fundamental polyhedron in the hyperbolic space, this portion of the field can hardly be said to have been more than lightly touched upon. (Compare the remarks on page 441, Vol. I.)

tain ternary and quaternary forms. This includes a large and important class of groups, among which the Picard group occurs as the reproducing group of a particular quaternary form. A considerable number of illustrative cases are worked out.

The third chapter deals with groups whose substitutions have coefficients which are integers within a given field of algebraic numbers.

Volume II, of which the first part only has appeared, is devoted to a study of the automorphic functions belonging to the groups treated in the first volume. By an automorphic function of an independent variable ζ is meant a function $\phi(\zeta)$ which is unaltered when ζ is transformed by any substitution

$$\zeta' = \frac{\alpha_k\zeta + \beta_k}{\gamma_k\zeta + \delta_k} \tag{6}$$

of a given group Γ; that is,

$$\phi\left(\frac{\alpha_k\zeta + \beta_k}{\gamma_k\zeta + \delta_k}\right) = \phi(\zeta).$$

It is further required that $\phi(\zeta)$ have no essential singularity at any point of the fundamental region of Γ, and that within this region it is uniform and without branching. Accordingly $\phi(\zeta)$ is expansible in the vicinity of an ordinary point ζ_0 in powers of $\zeta - \zeta_0$, and in powers of $1/\zeta$ in the vicinity of the infinite point.

If ζ_0 and ζ_0' are the fixed points of an elliptic substitution of period l, then since the corresponding substitution can be written in the form

$$\frac{\zeta' - \zeta_0}{\zeta' - \zeta_0'} = e^{\frac{2\pi i}{l}}\frac{\zeta - \zeta_0}{\zeta - \zeta_0'},$$

the function $\phi(\zeta)$ can be expanded in the vicinity of ζ_0 in a series of ascending powers of

$$\left(\frac{\zeta - \zeta_0}{\zeta - \zeta_0'}\right)^l.$$

Every parabolic substitution can be written in the form

$$\frac{1}{\zeta' - \zeta_0} = \frac{1}{\zeta - \zeta_0} + \gamma,$$

in which ζ_0 is the fixed point. In the vicinity of this point $\phi(\zeta)$ can therefore be expanded in a series of ascending powers of $e^{\frac{2\pi i}{\gamma}\cdot\frac{1}{\zeta-\zeta_0}}$.

The first step in the development of the theory of automorphic functions is to prove that such functions exist for every group Γ which is properly discontinuous in the ζ-plane. This is accomplished by means of Riemann's general existence theorem, the details of the proof following the methods of Schwarz and Neumann.

If $\zeta = \xi + i\eta$, it is first shown that an automorphic potential $u(\xi, \eta)$ can be found which satisfies prescribed boundary conditions, and is uniform and everywhere continuous in the fundamental region of Γ except at one prescribed point ζ_0 where it becomes discontinuous like the real part of $1/(\zeta - \zeta_0)$.

The conjugate potential v is defined by the integral

$$v(\zeta, \eta) = \int_{(\xi_0, \eta_0)}^{(\xi, \eta)} \left(\frac{\partial u}{\partial \xi}\, d\eta - \frac{\partial u}{\partial \eta}\, d\xi\right).$$

It is shown that dv has the automorphic character, and that hence v is reproduced by any substitution of the group with the addition of a constant. This constant is zero for groups of genus zero, and in that case the function

$$\phi(\zeta) = u(\xi, \eta) + iv(\xi, \eta)$$

is an automorphic function of ζ.

If $p > 0$, the expression $Z = u + iv$ behaves on the closed Riemann surface F, on which the fundamental region of Γ is depicted, like an elementary integral of the second kind. Hence, according to the well known Riemann method, if $Z_1, Z_2, \cdots, Z_\mu$ be μ such functions having different poles, constants $C_1, C_2, \cdots, C_\mu$ can be determined so that

$$C_1Z_1 + C_2Z_2 + \cdots + C_\mu Z_\mu$$

is reproduced unchanged when continued over any closed path in F. As a closed path in F corresponds to a path joining two congruent points in the ζ-plane, this means that ΣC_iZ_i is automorphic for the group Γ.

Among the properties of $\phi(\zeta)$ may be mentioned the following.

It takes any given value at μ points of the fundamental region P, μ being the number of poles of $\phi(\zeta)$. From this follows that the region P can be conformally represented on an ordinary μ-leaved Riemann surface F of genus p by means of the function $z = \phi(\zeta)$. Any automorphic function of ζ is accordingly an algebraic function on the Riemann surface F. On the other hand every algebraic function of z, $W[z(\zeta)]$, is an automorphic function of ζ, so that the totality of automorphic functions belonging to the group Γ coincides with the totality of algebraic functions on the surface F. It follows from this that between any two functions ϕ_1, ϕ_2 which are automorphic for the group Γ an algebraic relation exists, $G(\phi_1, \phi_2) = 0$. If this relation is irreducible, then every automorphic function belonging to the same group is rationally expressible in terms of ϕ_1 and ϕ_2. In particular, if $p = 0$, every such function is rationally expressible in terms of any function having a single pole in the fundamental region. Such a function is called a *principal function*. In general, a principal function is one which takes a given value the least possible number of times within the fundamental region.

It is evident from what precedes that every algebraic function on the Riemann surface F which is in general a many valued function of z, is single valued when expressed in terms of ζ. Every Riemann surface can be associated with a group Γ (in fact with more than one group), so that every algebraic function can be expressed as a uniform function. In the language of geometry, the coördinates of a point on any algebraic curve can be expressed as uniform (automorphic) functions of a variable parameter. To establish this result is one of the main problems of the book, or more explicitly stated: given any Riemann surface of genus p and n arbitrarily assigned points on it, $e_1, e_2, \cdots, e_n$, to investigate the existence of a function $\zeta(z)$ which branches at the given points and which can mediate the representation of the Riemann surface on a polygon suitable for the fundamental region of a limit circle group and having vertices corresponding to e_i which are fixed points for substitutions of periods $l_1, l_2, \cdots, l_n$, these being any integers > 1, or infinite.

Since $\phi(\zeta)$ takes any given value μ times in every polygon which is congruent to the fundamental region P, and since these polygons crowd together in infinite number about the limit points of the group, it follows that $\phi(\zeta)$ takes any given value an infinite number of times in the immediate vicinity of such a point which is accordingly a point of essential singularity for the function. If these limit points fill a curve, the limit curve, everywhere dense, this curve forms a natural boundary for the function over which it cannot be analytically continued.

Corresponding to the classification of the discontinuous groups, an analogous classification of the automorphic functions is given.

I. *Cyclic functions,* belonging to cyclic groups.

If the group is elliptic of period l, the principal function is the algebraic function $[(\zeta - \zeta_0)/(\zeta - \zeta_0')]^l$. If the group is parabolic, the principal function is $e^{\frac{2\pi i}{\gamma} \cdot \frac{1}{\zeta - \zeta_0}}$. If hyperbolic, or loxodromic, the automorphic functions are the elliptic functions of $\log [(\zeta - \zeta_0)/(\zeta - \zeta_0')]$.

II. *Elliptic functions.* The corresponding groups are the parabolic rotation groups.

III. *Fundamental circle functions.* These belong to groups whose limit points lie on a circle. If the circle is everywhere dense with fixed points, we have the special, but highly important, class of functions existing only within the circle and having the circumference for a natural boundary.

IV. *Automorphic functions in general, without fundamental circle.* This last and most extensive division of functions does not at present admit of precise classification owing to lack of knowledge of the non-rotation groups to which they belong. Three main subclasses are given, however, corresponding to the subdivisions of the groups mentioned above.

The inverse problem of determining the nature of ζ as a function of z in the Riemann surface F is next considered. When z describes a closed path in F, ζ is either unchanged or undergoes a linear transformation. It is infinitely many valued, and is accordingly called a polymorphic function. The important question arises as to whether in case any Riemann surface whatever is given, a polymorphic function exists, having the properties of the function ζ and capable of representing the Riemann surface on a polygon (or other region) suitable as the fundamental region for a group of linear substitutions. The full discussion of this problem is reserved for the remaining part of the volume which is not yet published, the author delaying on this point only to show that the polymorphic function ζ satisfies a differential equation of the third order obtained by equating the "Schwarzian derivative" of ζ to a certain function which is algebraic on the surface F.

From a consideration of automorphic functions of ζ we pass on in the second chapter to the consideration of forms belonging to groups of genus zero. The variable ζ is separated into the quotient of two variables ζ_1, ζ_2. Instead of the group Γ of substitutions of the form (6), the corresponding group of homogeneous substitutions

$$(7) \qquad \zeta_1' = \alpha_k\zeta_1 + \beta_k\zeta_2, \qquad \zeta_2' = \gamma_k\zeta_1 + \delta_k\zeta_2$$

is introduced. The course of the investigation is directed towards the construction of binary forms in ζ_1, ζ_2, having the automorphic character. This treatment of the problem from the point of view of binary forms is characteristic of the methods of Klein as previously employed in the Icosahedron theory and the Modular Functions.

An automorphic form is defined as a homogeneous, non-branching function of ζ_1, ξ_2 of dimension d (d being any rational number) which is reproduced multiplied by a constant μ when the variables describe any continuous series of admissible value such that the end values ζ_1', ζ_2' are congruent to the initial values with respect to the homogeneous group of unimodular substitutions (7).

That is,

$$\phi(\alpha_k\zeta_1 + \beta_k\zeta_2, \gamma_k\zeta_1 + \delta_k\zeta_2) = \mu_k\phi(\zeta_1, \zeta_2).$$

The quotient of two forms of like dimensions and the same multipliers is evidently an automorphic function of $\zeta_1/\zeta_2 = \zeta$.

Among the various forms that could be constructed for a given group ($p = 0$) two are of especial interest, the *principal form*, and the *prime form*. The former is defined by the equation

$$\phi_{-2}(\zeta_1, \zeta_2) = -\frac{1}{\zeta_2^2}\frac{dz}{d\zeta},$$

in which z is a principal function, that is, an automorphic function of ζ which takes a given value but once in the fundamental region. The principal form is absolutely invariant, is of dimension -2, has one pole of order 2 at the point where $z = \infty$, and vanishes of order $1 - 1/l_k$ at a vertex ϵ_k of the fundamental region which is a fixed point for a substitution of period l_k. It has no other poles or zeros.

If e_k is the value of z at $\zeta = \epsilon_k$, then $(z - e_k)$ vanishes of the first order at $\zeta = \epsilon_k$ and at all equivalent points: Hence the product

$$(8) \qquad \phi_{-2} \prod_{k=1}^{n} (z - e_k)^{-\left(1 - \frac{1}{l_k}\right)},$$

in which ϕ_{-2} is the principal form, does not vanish at the vertices, and is zero of order

$$\sum_k \left(1 - \frac{1}{l_k}\right) - 2 = -\frac{2}{\nu}$$

at the pole ζ_0 of ϕ_{-2}. Accordingly, if the expression (8) is raised to the power $\left(-\frac{\nu}{2}\right)$, the result vanishes of the first order at ζ_0. Such a form, denoted by $z_2(\zeta_1, \zeta_2)$, is of dimension ν in ζ_1, ζ_2, and is finite and different from zero at every point of the fundamental region except ζ_0. It is called a prime form.

The form

$$z_1(\zeta_1, \zeta_2) = z(\zeta) \cdot z_2(\zeta_1, \zeta_2)$$

is likewise a prime form whose zero point in the fundamental region coincides with that of z.

The formula $az_1 + bz_2$ defines a binary family of prime forms, all of which behave like z_2 with respect to the substitutions of the homogeneous group, and whose zero point varies with the parameter a/b.

If (z, e_k) denote the particular form of this family which vanishes at the fixed corner ϵ_k of the fundamental region, then the form

$$Z_k(\zeta_1, \zeta_2) = \sqrt[l_k]{(z, \epsilon_k)}$$

is a non-branching form for the group. It is called the *ground form* belonging to the corner ϵ_k. In case of an elliptic point l_k is the period of the substitution, and for a parabolic point l_k may be any positive integer.

The effect of the substitutions of the homogeneous group upon any form ϕ_d is next determined. This is accomplished by calculating the multipliers μ_k which arise from the generating substitutions. From this set of multipliers, called a multiplier system, the multiplier for any other operation of the group is obtained by a very simple formula.

The number of theoretically possible multiplier systems having been determined, it is then proved that automorphic forms of any allowable dimension d exist for every such system. This is done by first showing that every automorphic form can be expressed as a product of prime forms and ground forms, and that this product can be so chosen that the system of multipliers will coincide with any given possible system. In particular, the forms with the multiplier system $\mu_k = 1$ exist for every group Γ and for every integer dimension d. They are absolutely unchanged by substitutions of the group, and for this reason are called *proper* automorphic forms.

The variables ζ_1, ζ_2 when regarded as depending on z_1, z_2 are polymorphic forms in the latter. On multiplying ζ_1 and ζ_2 by certain forms in z_1, z_2 polymorphic forms are obtained which are of zero dimension and behave like ζ_1, ζ_2 with respect to substitutions of the group. Analytic expressions for these various forms are obtained, and differential equations of the second order which they satisfy are deduced.

Some of the foregoing results are illustrated by means of the hypergeometric function.

The third chapter treats of the Poincaré series with a detailed consideration of the case $p = 0$. This subject, which is one of the most important and fundamental in the entire theory of automorphic functions, is handled in a very felicitous and attractive manner.

"Poincaré series" is the name given in the present work to the series

$$(9) \qquad \phi(\zeta_1, \zeta_2) = \sum_k \mu_k^{-1} H(\alpha_k\zeta_1 + \beta_k\zeta_2, \gamma_k\zeta_1 + \delta_k\zeta_2),$$

in which $H(\zeta_1, \zeta_2)$ is any rational homogeneous function of ζ_1, ζ_2, and μ_k is the multiplier, of any allowable system, corresponding to the transformation (7). The function ϕ thus defined is an automorphic form for the given group with the multiplier system μ_k.

If the above series be divided by ζ_2^d, d being the dimension of ϕ, and the multiplier system $\mu = 1$ taken, the resulting formula

$$\Theta(\zeta) = \sum_k H\left(\frac{\alpha_k\zeta + \beta_k}{\gamma_k\zeta + \delta_k}\right)(\gamma_k\zeta + \delta_k)^d,$$

with d a negative even integer, is the "theta" series introduced by Poincaré, and is the principal element out of which he constructs the analytic theory of the automorphic functions. This function is reproduced, multiplied by a factor, by any substitution of the group; that is,

$$\Theta\left(\frac{\alpha_i\zeta+\beta_i}{\gamma_i\zeta+\delta_i}\right)=(\gamma_i\zeta+\delta_i)^{-d}\Theta(\zeta),$$

while the corresponding form ϕ is entirely unchanged.

The series (9) is proved by two methods, both due to Poincaré, to be convergent. The first proof establishes the convergence for a group with one limit curve, or an infinite number of them, and for all dimensions $d \leqq -4$. This is the most general case and includes all others.

Under what conditions the Poincaré series is convergent for values of d greater than -4 is a problem not yet completely solved, but the case $d=-2$ receives somewhat extensive consideration. It is shown that for this value of d the series is no longer absolutely convergent whenever the group has one or more limit curves; but that for fundamental circle groups which are not limit circle groups, and for certain other groups without limit curves, the series is unconditionally and uniformly convergent. These results, which are largely due to the investigations of Burnside and Schottky, lead the authors to suggest with Burnside that the theorem is possibly true for all groups without limit curves. It is worthy of remark, also, that groups exist (for every genus p) for which the series of dimension $d=-1$ is absolutely convergent.

The second proof is given for the case of a fundamental circle group, and establishes the convergence of the series (9) for all dimensions d less than -2.

The quotient of two Poincaré series of like dimension d and with the same multiplier system is an automorphic function of $\zeta=\zeta_1/\zeta_2$. Thus the Poincaré series affords a simple and elegant formula for the construction of functions belonging to the group.

One serious obstacle in the way of using this series arises from the possibility of its identical vanishing. That this possibility is not an imaginary one is shown by proving that identically vanishing series exist, and in infinite number for every dimension d. The difficulty is avoided by the construction of series having one or more poles in the fundamental region. This is effected by assigning poles to the function H which is used in generating the series. The case of a single pole in the fundamental region being the simplest one is considered in detail. It is proved that such a series can always be constructed, and the details of the method, and the actual form of the series are given except in the case of absolute automorphic forms of dimension -2 which must have at least two poles. It is shown that the pole can be taken at any point in the polygon net, the parabolic fixed points alone excepted.

The one-pole series is now placed in the foreground. On the introduction of a suitable multiplying constant, it becomes discontinuous at the pole $\zeta_1 = \xi_1$, $\zeta_2 = \xi_2$ like $1/(\zeta_1\xi_2 - \zeta_2\zeta_1)$. The series so normalized is called an *elementary form* and is denoted by $\Omega(\zeta_1, \zeta_2; \xi_1, \xi_2)$. It is worthy of notice that this elementary form, when regarded as depending on ξ_1, ξ_2, is of dimension $-d-2$, and in its manner of becoming discontinuous behaves like an automorphic form with inverse multiplier system μ_k^{-1}. In order that Ω may be an automorphic form in ξ_1, ξ_2 it is necessary that, when expressed as a product of prime forms and ground forms, it be of dimension -1 in the former. It also has the same property when regarded as a form in ζ_1, ζ_2, which fact leads to the expression of Ω in this case by a very elegant formula [(12), page 201].

Except for the special case just mentioned, Ω is not automorphic in ξ_1, ξ_2. It has, however, certain properties as a function of these variables, which lead to the important conclusion that every automorphic form which vanishes in the parabolic points can be expressed in the form of a Poincaré series provided the dimension d is one for which that series is convergent. That automorphic forms exist which are not expressible as Poincaré series is shown by the occurrence of such a case for $d = -1$ in the modular group.

This result is further generalized by showing that every automorphic form with arbitrary poles none of which occur at a parabolic point, is expressible as a Poincaré series plus an integral automorphic form.

The fourth and last chapter generalizes the results to the case $p > 0$. As in this case the automorphic functions are no longer expressible in terms of a principal function z, the simple prime form for $p = 0$ must be replaced by the transcendental prime form which enters into the theory of abelian functions.

The Ritter prime form introduced here, while not so simple an analytic expression as the Klein prime form, has the advantage over the latter of not vanishing at the branch points of the Riemann surface F.

Klein's prime form $\Omega(x, y)$ is defined as the limit of the expression

$$\sqrt{-(x, dx)(y, dy)\, e^{-\Pi_{x, y}^{x+dx,\, y+dy}}}$$

for $dx = 0$, $dy = 0$. Here $\Pi_{x, y}^{\xi, \eta}$ is the normal integral of the third kind on the m-leaved Riemann surface, while (x, dx) is written for brevity to represent the homogeneous differential expression $x_1 dx_2 - x_2 dx_1$.

Ritter's prime form $P(z, e)$ is

$$\Omega(z, e) \sqrt[m]{\frac{z_2}{\Omega(z, \infty_1)\, \Omega(z, \infty_2) \cdots \Omega(z, \infty_m)}}$$

in which e is any point of the Riemann surface and $\Omega(z, \infty_i)$ is Klein's prime form with the path of integration extending to the point at infinity in the ith sheet of the surface, while z_2 is one of the homogeneous variables into which $z = z_1/z_2$ is separated. The prime form thus defined is everywhere continuous and has only a single zero of the first order at $z = e$. It is, however, in the periodic property of this prime form with respect to substitutions of the group Γ that its superiority over the Klein form is especially marked, for while the latter is reproduced multiplied by a function of z, the former has a multiplier independent of z.

Expressions are next obtained in terms of the Ritter prime form for the polymorphic forms ζ_1, ζ_2 whose quotient gives the variable ζ in terms of which z is an automorphic function.

Just as in the case $p = 0$, the variable ζ as a function of z satisfies a differential equation of the third order obtained by equating the Schwarzian derivative to a certain function R which is rational and algebraic on the Riemann surface, and which depends on $n + 3p - 3$ arbitrary constants, the accessory parameters of the differential equation. The form of the function R is determined for three cases of special interest, viz:

1. When the group Γ is of character $(1, n)$.
2. When it is hyperelliptic of character $(p, 0)$.
3. For the general case of character $(3, 0)$.

The polymorphic forms ζ_1, ζ_2 satisfy the differential equation of the second order

$$\frac{d^2F}{dx^2} + RF = 0,$$

in which R is the function occurring in the differential equation of the third order for ζ.

The general multiplicative forms, that is, the automorphic forms which are reproduced with multiplying constants by the operations of the group Γ, are now taken under consideration, the main result arrived at being the theorem that such forms always exist for every integer dimension d and for every theoretically possible multiplier system. Every form ϕ of this kind is expressible as a product of ground and prime forms giving the zeros of ϕ, divided by a product of prime forms giving the poles of ϕ. As in the case $p = 0$, the ground form is a properly chosen root of a prime form which vanishes at one of the fixed corners of the fundamental polygon. By prime form is here meant the product of the Ritter form by an exponential factor whose exponent is an abelian integral of the first kind on the Riemann surface, such an exponential factor being the most general expression for a multiplicative form without poles or zeros.

Among the automorphic forms belonging to a group of genus p those of dimension -2 and multiplier system 1 which are free from poles have particular interest. Denoting such a form by $\Phi_{-2}(\zeta_1, \zeta_2)$ it is observed that the integral $\int \Phi_{-2}(\zeta_1, \zeta_2)(\zeta, d\zeta)$ is of dimension zero and hence a function of ζ (and therefore of z) which is everywhere finite on the Riemann surface F. It is accordingly an abelian integral of the first kind. As there are p linearly independent integrals of the first kind it follows that the same number of linearly independent functions Φ_{-2} exist. The number of linearly independent forms ϕ of given dimension and multiplier system is $t - p + \sigma + 1$, in which t is the number of zeros of ϕ, and σ is the number of linearly independent forms Φ_{-2} which vanish in these t zero points. This is clearly only another form of the Riemann-Roch theorem.

The expression of automorphic forms by means of the Poincaré series next demands attention. It is to be remarked first of all that in order to insure the convergence of this series, the multipliers μ_k were assumed unimodular. If a form ϕ' has multipliers which are not unimodular, it can be expressed in the form

$$\phi' = e^{\Sigma c_i w_i}\phi,$$

in which ϕ is a form with unimodular multipliers and the w_i are normal integrals of the first kind on the Riemann surface F. A generator of Γ which corresponds to a crossing of a canonical period path in F will then reproduce ϕ' with a multiplier $e^{c_i p_i}\mu_i$, p_i and μ_i being the corresponding moduli of periodicity of w_i and the multiplier of ϕ respectively. By a proper choice of the c_i it is evident that any desired multiplier system may be assigned to ϕ'.

In order that the form ϕ be expressible as a Poincaré series it is further necessary that it vanish in the parabolic fixed points. After a somewhat lengthy, but interesting analysis, a conclusion is reached similar to that in case $p = 0$, namely — every unimultiplicative form of dimension d satisfying the convergence condition, which vanishes in the parabolic points, and only such a form, is expressible as a Poincaré series; and every automorphic form with arbitrary poles, none of which occur at parabolic points, is expressible as a Poincaré series plus an integral automorphic form.

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LORIA'S SPECIAL PLANE CURVES.

Spezielle algebraische und transcendente ebene Curven, Theorie und Geschichte. Von Dr. GINO LORIA. Autorisierte, nach dem italienischen Manuscript bearbeitete deutsche Ausgabe von FRITZ SCHÜTTE. Leipzig, B. G. Teubner, 1902. 8vo., xxi + 744 pp. + 17 plates containing 174 figures.

ABOUT ten years ago the Royal Academy of Sciences at Madrid offered its triennial prize to be awarded upon the thirty-first of December, 1894, for "An ordered account of all curves of any kind which had received special names, and a further short account of their form, their equations, and their inventors." To this prodigious question no response seems to have come. Three years later the question was repeated. Professor Loria presented his researches, which were received