includes many corrections not given by Schering in his appendix to Gauss's volume, or by Perott.

	For.	Read.			For.	Read.
468	*Reg.*	*Irr.* 2	2331		*Reg.*	*Irr.* 2
485	IV 4	IV 5	2196		*Reg.*	*Irr.* 2
544	*Reg.*	*Irr.* 2	2180		*Reg.*	*Irr.* 2
547	*Reg.*	*Irr.* 3	2304		*Reg.*	*Irr.* 2
557	II 11	II 13	2320		*Reg.*	*Irr.* 2
647	I 25	I 23	2624		*3*	*2*
894	IV 6	IV 7	2336		*Reg.*	*Irr.* 2
931	*Reg.*	*Irr.* 3	2900		*Reg.*	*Irr.* 2
933	IV 3	IV 4	2188		*Reg.*	*Irr.* 3
972	*Reg.*	*Irr.* 3	2085		VIII 5	VIII 4
993	IV 4	IV 3	2096	in IV 6	2096	2097
1116	IV 9	IV 6	2204		IV 11	IV 13
1261	II 10	IV 5	2221	in IV 9	2221	2224
1367	I 27	I 25	2376	in IV 12	2376	2366
1396	IV 7	II 14	2448		*Irr.* 2	*Reg.*
1508	*Reg.*	*Irr.* 2	6032		*Reg.*	*Irr.* 2
1598	*Reg.*	*Irr.* 2	6068		*Reg.*	*Irr.* 2
1660	IV 4	Omit	6084		*Reg.*	*Irr.* 2
1683	II 9	II 6	6148		*Reg.*	*Irr.* 2
1700	———	IV 12	6176		*Reg.*	*Irr.* 2
1701	*Reg.*	*Irr.* 3	9104		*Irr.* 2	*Reg.*
1725	*Reg.*	*Irr.* 2	9108		*Reg.*	*Irr.* 2
1796	IV 10	II 20	9156		*Reg.*	*Irr.* 2
1836	*Reg.*	*Irr.* 3	9216		*Reg.*	*Irr.* 2
1872	*Reg.*	*Irr.* 2	9324		*Reg.*	*Irr.* 2
1937	———	IV 12	9513		*Irr.* 2	*Reg.*
1982	IV 12	Omit	9554		*Reg.*	*Irr.* 2
1940	IV 8	IV 10	6075		*Irr.* 3	*Irr.* 9

THE LOGARITHM AS A DIRECT FUNCTION.

BY DR. EMORY MCCLINTOCK.

(Read before the American Mathematical Society, February 28, 1903.)

IN a paper of the same title published in the *Annals of Mathematics* for January, 1903, Mr. J. W. Bradshaw defines $\log x$ as a direct function of x, namely,

$$\log x = \int_1^x x^{-1} dx.$$

Attention being thus drawn to the subject, I think the time opportune to repeat and amplify a proposition of my own for the same general purpose.

In 1879 (*American Journal of Mathematics*, II, 101, etc.) I spoke of "the difficulty of comprehending logarithms," and quoted De Morgan's dictum that "the only definition of log x used in analysis is y, where $e^y = x$." After discussing this definition I said, "Another and, when duly weighed, most satisfactory definition may be derived from any one of an unlimited number of vanishing fractions, special cases of the general form $\log x = h^{-1}(x^{(1-a)h} - x^{-ah})$, where h is infinitely reduced. * * * This fraction is doubtless novel, though one case of it, where $a = 0$, is known. Even that case has not, I presume, been suggested heretofore as a definition. * * * The various theorems pertaining to logarithms may be derived with the utmost facility by the aid of these vanishing-fraction definitions. Thus, if $a = 0$, we have by expansion

$$\log(1 + x) = \frac{(1 + x)^h - 1}{h}[h = 0] = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \cdots."$$

To develop this proposition more fully, let us consider the function $y = h^{-1}(x^h - 1)$. Let k be positive, and, first, let $h = k$. When $x = 1$, $y = 0$; when $x = \infty$, $y = \infty$; and as x increases continuously from 1 towards ∞, there is one and only one corresponding value of y, which increases accordingly from 0 towards ∞. Secondly, let $h = -k$. Here again, when $x = 1$, $y = 0$; and, in the function $y = k^{-1}(1 - x^{-k})$, as x increases continuously from 1 towards ∞, there is one and only one corresponding value of y, which increases accordingly from 0 to k^{-1}, a limit which tends towards ∞ if k tends towards 0. When $h = -k$, $y = k^{-1}(x^k - 1)x^{-k}$, which differs only by the factor x^{-k} from the value of y when $h = k$. The smaller k is taken, the nearer this factor is to 1, so that the limit of the value of y, for $h = 0$, is the same whether h is positive or negative, while

$$y_{[h<0]} < \lim_{h=0} y < y_{[h>0]}.$$

The limit is therefore a 1 to 1 function of $x > 1$. When $x = 1$, the limit is 0. When $0 < x < 1$, let $x = u^{-1}$, where $u > 1$; then we have $h^{-1}(x^h - 1) = h^{-1}(1 - u^h)x^h$, and, since $\lim_{h=0} x^h = 1$, $\lim_{h=0} h^{-1}(x^h - 1) = -\lim_{h=0} h^{-1}(u^h - 1)$.

Let us define the logarithm of x (positive) as $\lim_{h=0} h^{-1}(x^h - 1)$, and denote it by $\log x$. We have just found that $\log(x^{-1}) = -\log x$. Since $\lim_{h=0} b^h = 1$,

$$\log a = \lim_{h=0} h^{-1}(a^h b^h - b^h) = \log(ab) - \log b.$$

This is the chief property of logarithms. Hence, $\log(a^2) = 2\log a$, $\log(a^n) = n\log a$, and if $b = a^n$, lo $(b^{1/n}) = 1/n \log b$, which might be used, as by Mr. Bradshaw from another definition of $\log x$, to show that for every positive number b there exists one and only one positive nth root. Here n is a whole number. It follows that $\log b^{m/n} = m \log(b^{1/n}) = m/n \log b$. If we take n incommensurable, let $a = b^m$, where m is an integer, $b = 1 + c$, and $-1 < c < 1$. Employing the binomial expansion,

$$\begin{aligned}\log(b^n) = \lim_{h=0} h^{-1}[(1+c)^{nh} - 1] &= n(c - \tfrac{1}{2}c^2 + \tfrac{1}{3}c^3 - \cdots) \\ &= n\log(1+c) = n\log b.\end{aligned}$$

Hence

$$\log(a^n) = \log(b^{mn}) = m\log(b^n) = nm\log b = n\log a.$$

That the continuous function $\log x$ has a continuous derivative x^{-1} may be shown thus, with $\Delta x < x$:

$$\frac{d \log x}{dx} = \lim_{\Delta x=0} \lim_{h=0} h^{-1}(\Delta x)^{-1}[(x+\Delta x)^h - x^h].$$

If we expand the part within the brackets and divide the resulting series throughout by $h\Delta x$, we have

$$\frac{d\log x}{dx} = \lim_{\Delta x=0} \lim_{h=0} \left[x^{h-1} + \tfrac{1}{2}(h-1)(\Delta x)x^{h-2} + \frac{1}{2\cdot 3}(h-1)(h-2)(\Delta x)^2 x^{h-3} + \cdots \right].$$

If we first put $h = 0$, the part within brackets becomes $x^{-1} - \frac{1}{2}(\Delta x)x^{-2} + \frac{1}{3}(\Delta x)^2 x^{-3} - \cdots$, which is x^{-1} when $\Delta x = 0$. If we first put $\Delta x = 0$, the part within brackets becomes x^{h-1}, which is x^{-1} when $h = 0$.