enjoyed the loving attention of one who sacrificed a large part of his scientific activity in erecting a lasting monument to the memory of Germany's princeps mathematicorum.

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ANALYTIC PROJECTIVE GEOMETRY.

Lehrbuch der analytischen Geometrie in homogenen Koordinaten. Von WILHELM KILLING. 2 Teile, 8vo. Paderborn, F. Schöningh. Teil 1: *Die ebene Geometrie,* 1900. xiii + 220 pp. Teil 2: *Die Geometrie des Raumes,* 1901. viii + 361 pp.

THE first and perhaps the most noticeable feature of Professor Killing's text-book on the analytic geometry of homogeneous coördinates is the elaborately methodical arrangement. The line of march has been laid out with extreme care. Notice, for example, the titles and page numbers of the first few sections of the two volumes:

In the plane, Vol. 1.	In space, Vol. 2.
§ 1. Theory of cross ratio, p. 1.	§ 1. Division of dihedral angles, p. 1.
§ 2. The coördinate triangle, p. 8.	§ 2. The coördinate tetrahedron, p. 7.
§ 3. The straight line, p. 13.	§ 3. The straight line and plane, p. 13.
§ 4. The perpendiculars dropped upon a straight line from the vertices of the coördinate triangle, p. 19.	§ 4. The perpendiculars dropped upon a plane from the vertices of the coördinate tetrahedron, p. 21.
§ 5. The most general trimetric coördinates, p. 25.	§ 5. The most general tetrahedral coördinates, p. 29.
§ 6. The ratios of the coördinates as cross ratios, p. 37.	§ 6. The elements at infinity, p. 38.
§ 8. The elements at infinity, p. 48.	§ 7. The quotients of the coördinates as cross ratios, p. 44.
Total number of pages, 49.	Total number of pages, 52.

There is no necessity of stopping the parallel column here. It might run, with very few exceptions, to the end of the two volumes. Each closes with the treatment of confocal quadrics; the treatment in space is, of course, more elaborate than that required in the plane, but otherwise exactly parallel to it. The differences between the texts of Volume 1 and Volume 2 in the first fifty pages are of about the same order as the difference in the sectional headings quoted above. The changes are of two kinds: Those that are necessitated by the subject — the substitution of "line and plane" for "line," of four variables or four equations for three; and those changes which seem unnecessary — the substitution of the word "quotient" for "ratio" in the sectional headings 6 and 7, and the rearrangement of the order of the sections where it is of no particular importance.

This is carrying method to a painful extreme. There can be little excuse for spending 52 pages of Volume 2 upon those sections which correspond exactly to the 49 pages of Volume 1. The author might, at least, have economized in Volume 2 enough to save the three extra pages needed by the constantly recurring fourth variable and fourth equation. One could throw away the first part and commence on the second without experiencing any particular difficulty. So much elaboration of method is quite unnecessary. The book is well named a *Lehrbuch,* and one can almost hear, almost see the *Lehrmeister* as he reaches the second part say to his pupils: "I have explained the first volume with all care. Here is the second. Read it. It is like the first volume. Toward the end there are a few new things — no, not new, only a little less familiar, a little harder to generalize. If you find difficulty, wake me." And it would be a worthless student who need wake his master. More likely he would himself fall asleep over the monotony, the lack of opportunities to employ his wits and imagination.

Apart from this feature and from its antiquated typography, with spaced letters instead of italics and formulæ in the same font as the text, the book is good. The general plan is excellent; the scope has been chosen with judgment. The subjects treated are not too many nor too few. The methods of analysis are for the most part neat. The author has exercised great care in treating the matter of sign, which is so confusing to the beginner in connection with the coördinate triangle. Although the book is confessedly analytic there is often a glimmer, unfortunately only a glimmer, of truly geometric reasoning. The large numbers of exercises distributed through the text, some at the end of each section, have been selected with especial care, assorted, and graded. For this reason the book must be more than usually helpful to a teacher, especially a young and inexperienced teacher, in preparing a course on projective geometry. To a student, who works without a teacher, who wishes to supplement a meager knowledge or even to gain from the beginning some idea of projective geometry, this book with its careful explanations and its problems will prove a great boon. It offers but little difficulty. It is so easy, elementary and slow; but withal, when one has finished it, so inclusive.

A comparison of the book under review with Duporcq's Premiers principes de géométrie moderne * is very instructive and interesting. Both treat projective geometry, though Duporcq concludes with a chapter on other transformations. But the methods are entirely different. The one is methodical, the other not at all; the one painstakingly accurate in demonstration, the other often hasty and fond of relying upon the sweep of intuition; the one analytic, the other geometric with only a background of analysis, like Chasles; the one assuring, the other stimulating. The fact is that a student of Killing would know that he had at his command a perfectly definite method. On meeting a problem he could work out a solution and probably put it in good form. Yet he very likely would not be one whit a geometer, but merely an analyst. The student of Duporcq could hardly get along without a teacher to help him. He would be quite unable to say whether he had a method at his command or not. He would know many theorems, and if he came to a problem he might or might not solve it. If he solved it he would not know why he had succeeded, nor in what section of geometry to place it. Yet he would think geometrically. The two books are complements of each other. The student of either would find in the other just what he lacked. If he were studying alone, he had best begin with Killing; if with a teacher, with Duporcq.

The place of trilinear and tetrahedral coördinates, the place of projective geometry in general in our mathematical education, has changed considerably since the first appearance of

* Reviewed in the BULLETIN, vol. 6, p. 254.

those classics — Salmon's Conic Sections and Clebsch's Vorlesungen über Geometrie — which, with their various translations into other languages, have done so much to influence the subsequent treatment of projective geometry. Thirty or forty years ago the group of projective transformations, then scarcely recognized as a group, was playing a well-nigh all-important rôle in mathematics owing to its connection with the so-called higher algebra. The vast theory of invariants — invariants of the projective group — was in building. Cayley and Clebsch were doing their great work. The particular enthusiasm of that day has passed by. As one looks over the announcements of the different universities he looks almost in vain for a course upon this theory of invariants. The theory of invariants now means a part of Lie's theory of groups.

The projective transformation itself seems less important. Inversion with its applications to the theory of functions, to potential, to all problems of conformality demands constantly more attention. Moreover there is the vast body of transformations brought to our notice by Lie under the name of contact transformations, of which even a single one, the line-sphere transformation, is of great importance. Thus even if we look only at the field of geometry we must grant that the projective transformation cannot hope to occupy so much of one's time and thoughts as formerly. And when we take into consideration other fields which have been springing up so rapidly and which contain scarcely the slightest reference to projective geometry, we may well wonder whether or not this subject deserves any special consideration, whether we are not giving it too much attention, whether in reality we need it at all except as a simple example of a group of transformations with very interesting properties.

Certain it is that the elaborate volumes on trilinear and tetrahedral coördinates are less attractive now. Room and time must be left for circle and sphere and line (in space) coördinates. Certain it is, too, that now, after the passing of the formal invariant theory of the projective group, projective geometry is taking its place beside other important geometries and we must in every way try to present its essentials as rapidly as possible and without any hope of elaborating its details. The other geometries cannot longer be neglected.

That, however, which will always render projective geometry the best introduction to other geometries is its geometric simplicity and elegance. Its conceptions, its transformations are more easily visualized. The figures with which it deals are of the simplest. The methods of treatment have been so developed that one may proceed analytically or synthetically and may thus acquire at the same time both analytic and synthetic facility. To mix these two methods is distasteful to some persons. But the fact is that many theorems are more easily proved synthetically, many others analytically. In the latter case one must needs commence at the beginning with a coördinate system and work up; in the former one starts at least half way up, resting upon a large basis of known theorems. Each method has its advantages. It is still better to mingle the methods, changing from one to the other whenever the change illuminates the problem or facilitates its solution. In other words, nowadays, when a time and space limit is imposed upon us, we must at first leave aside puristic considerations of all analysis or all synthesis, and must content ourselves with developing both the analytic and synthetic methods in less time than formerly was allotted to each.

For this purpose many properties of trilinear coördinates become useless. It is of little value to know that the trilinear coördinates of a point are proportional to constant multiples of the perpendicular distances of the point from the sides of the triangle of reference. The property is not projective. On the other hand it is important to know that the ratios of the coördinates are cross ratios at the vertices or upon the sides of the triangle of reference. This property is projective and in fact assures us that the coördinate system chosen is the most general.

To shorten further and simplify the analytic presentation of projective geometry it is necessary carefully to isolate the difficulties which appear in the subject from the student's standpoint in passing from cartesian geometry. These difficulties seem to be three: The question of infinity, the fact that homogeneous instead of non-homogeneous coördinates are used, the idea and use of line coördinates in the plane or plane coördinates in space are very real difficulties. Each is a new idea to the student. They occur again and again in other geometries than projective. Beyond these three there is scarcely any other difficulty unless it be the use of abridged notation in avoiding the solution of equations. For in projective geometry as in other higher geometries one constantly aims at using expressions like $u + \lambda v = 0$, $uv = 0$, determinants, etc., as much as possible to avoid tedious analysis. This can be done, although it usually is not done, in ordinary cartesian geometry.

For the past two years it has fallen to my lot to teach plane projective geometry to a class of students who know no other geometry than the euclidean and the elements of the cartesian. The allowance of time has been a half-year, forty-five lessons. This is short — perhaps too short. But when we consider the vast number of subjects, the calculus, mechanics, geometry, algebra, and perhaps physics, which demand the attention of an intermediate student of mathematics, say a junior in college, one is perhaps willing to grant that projective geometry cannot claim more. As many others may encounter similar problems in instruction the account of my experience, which comes in naturally in connection with my criticisms of Killing's Geometry, may be pardoned.

To have followed a book such as Killing's or Salmon-Fiedler's would have been to give the student not more than a bare working knowledge of trilinear coördinates of point and line. To educate his geometric insight to any great extent would have been impossible. To have followed exclusively the synthetic methods would have been to leave the student not very well grounded in them, and absolutely without the basis of analysis. It was evidently necessary, and only fair, to combine the synthetic and analytic methods.

To get the student back to where he was when he finished his Euclid, to get him again to thinking about geometric objects and geometric methods, to relieve him of the idea that geometry beyond Euclid consists merely of sets of algebraic equations of the first and second degree, it was necessary to begin with pure synthetic geometry. The subjects treated were:* Harmonic elements; the principle of duality; the elements at infinity; projective relations upon a line; involution; collineation and correlation in the plane; polarity; the conic; projective relations between ranges or pencils upon a conic. This much insures that the student thinks geometrically, that he has settled his mind about infinity and that he can solve a tolerably large number of problems by means of the fundamental theorems which he has learned.

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*See my paper on "The synthetic treatment of conics" in the BULLETIN, 1903, p. 248.

The analytic treatment is then conducted as follows: Abridged notation; the introduction of homogeneous rectangular coördinates $x : y : t$, defined by $X = x : t$, $Y = y : t$; line coördinates defined as the negative reciprocals of the intercepts of a line upon the axes X and Y; homogeneous line coördinates; the cross ratio, and its independence of projection; the fact that the ratios of the homogeneous point or line coördinates are cross ratios upon the axes or at infinity; the projection of the rectangular axes of reference and the line at infinity and the unit point into an arbitrary triangle and arbitrary point; the point and line coördinates which result; practice in the use of these coördinates. It will be seen that the difficulties of abridged notation, homogeneous coördinates and line coördinates were each treated before definitely leaving the familiar system of rectangular cartesian coördinates. It will be seen furthermore that trilinear coördinates, together with their triangle and unit point of reference, were introduced as a mere projection of customary coördinates. The two facts which allow this are the theory of collineations as explained in synthetic geometry and the invariance of the cross ratio. It is seen also that the analysis in trilinear coördinates is identical with the analysis in rectangular homogeneous coördinates, if the triangle of reference and the figure to be investigated be both projected back on the plane of X and Y. This numerical identity of the calculations in the two cases is a great help in giving the student confidence in his analysis. It is very much more valuable than the ordinary method of deducing the coördinates by the method of perpendiculars. It also avoids the necessity of the proof that the linear equation represents a straight line or point.

Generally after this method of presentation there are still a few lectures remaining for the purpose of review, to say a few words about quadratic and Cremona transformations, to take up the circular points and focal properties, or to outline the generalization to space. The student has at any rate a working knowledge of synthetic and analytic methods in the plane.

To any one who wishes to work up a course such as that outlined, the following three books: Killing's Lehrbuch der analytischen Geometrie, for its analysis; Enriques's Geometria proiettiva, for its synthetic methods; and Duporcq's Premiers principes de géométrie moderne, for its intermediary liveliness and its exhibition of the relations between metric and projective geometry, form a library which is quite sufficient and one may almost say necessary. There may be other books as good; but for this particular purpose these are not easily improved upon.

EDWIN BIDWELL WILSON.

ÉCOLE NORMALE SUPÉRIEURE,
PARIS, FRANCE,
December 5, 1902.

SHORTER NOTICES.

Urkunden zur Geschichte der Mathematik im Mittelalter und der Renaissance. By M. CURTZE. Zweiter Theil. *Abhandlungen zur Geschichte der mathematischen Wissenschaften,* XIII Heft. Leipzig, Teubner, 1902. 292 pp. 14 marks.

Abhandlungen zur Geschichte der mathematischen Wissenschaften, XIV Heft, Leipzig, Teubner, 1902. 338 pp. 16 marks.

THE first part of Curtze's Urkunden, forming the twelfth volume of the Abhandlungen, has been reviewed in the BULLETIN * so recently that it is quite surprising to find two new volumes of the series already published. Indeed no better evidence of the present revival of interest in the history of mathematics can be found than is seen in the encouragement recently given to this series founded a quarter of a century ago by Professor M. Cantor. The publication of the first seven volumes extended through a period of nineteen years, while the last seven, including the two under review, have appeared since 1897.

The second part of Herr Curtze's Urkunden is devoted to two interesting manuscripts, one the Practica Geometriæ of Leonardo Mainardi of Cremona, and the other the algebra of Initius Algebras. The first, which also bears the title Leonardi Cremonensis Artis Metrice Practice Compilatio, is a transcript, with German translation, from an Italian codex in the Venetian dialect in the university library at Göttingen. This codex is not unique, for Prince Boncompagni had two Latin manuscripts of the same work; but not only has it never before been published, but Leonardo Mainardi has been practically unknown to historians of mathematics. It consists of fifty folios, of which the first twenty-nine and the last fourteen are here

* Vol. 9, p. 123, Nov., 1902.