THE ABSTRACT GROUP G SIMPLY ISOMORPHIC WITH THE ALTERNATING GROUP ON SIX LETTERS.

BY PROFESSOR L. E. DICKSON.

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1. A SLIGHT correction of a theorem due to De Séguier * leads to the result that G is generated by three operators a, b, c, subject only to the relations

$$(1) \qquad a^2 = I, \quad b^4 = I, \quad c^3 = I, \quad (ac)^3 = I,$$

$$(2) \qquad (ab^{-1}ab)^3 = I, \quad (ab^{-2}ab^2)^2 = I,$$

$$(3) \qquad (cb^{-1}ab)^2 = I, \quad (cb^{-2}ab^2)^2 = I.$$

But these generators are not independent, since

$$(4) \qquad a = cb^{-1}cbc.$$

A simple verification of (4) results from the correspondence

$$a \sim (12)(34), \qquad b \sim (12)(3456), \qquad c \sim (123)$$

between the generators of the simply isomorphic groups.

It is shown in this section that G is generated by the two operators b and c, subject to the complete set of generational relations

$$(5) \qquad b^4 = I, \quad c^3 = I, \quad (b^{-1}cbc^{-1})^2 = I, \quad (b^2c)^4 = I.$$

These relations follow from (1), (2), (3); for, by the above correspondence, $b^{-1}cbc^{-1} \sim (14)(23)$, $b^2c \sim (1235)(46)$.

If a be defined by (4), *relations* (1), (2), (3) *follow from* (5).

$$a^2 = cb^{-1}cbc^{-1}b^{-1}cbc = c(b^{-1}cbc^{-1})^2c^{-1} = I,$$

$$(ac)^3 = cb^{-1}c^3bc^{-1} = I.$$

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* *Journal de Math.*, 1902, p. 262. For $y = 2, \ldots, n-3$ in his formula (6), should stand $y = 1, \ldots, n-4$.

As an auxiliary result, we note that

(6) $$c(bcb^{-1}cb) = (bcb^{-1}cb)c.$$

The condition (6) may be given the successive forms

$$cbcb^{-1} \cdot cbc^{-1}b^{-1} \cdot c^{-1}bc^{-1}b^{-1} = I,$$

$$cbcb^{-1} \cdot bcb^{-1}c^{-1} \; c^{-1}bc^{-1}b^{-1} = (cbc^{-1}b^{-1})^2 = I \qquad [\text{by } (5_3)].$$

Since (6) may be written $cbac^{-1} = ba$, we have

(7) $$c \cdot ba = ba \cdot c.$$

In view of (6) and (5_3), we get

(8) $$(cb^{-1}cb)^3 = cb^{-1} \cdot bcb^{-1}cb \;\; c^2b^{-1}cb = I.$$

To verify (3_1) we note that, by (8),

$$cb^{-1}ab = cb^{-1} \cdot cb^{-1}cbc \;\; b = cb^{-2}c^{-1}bc^{-1}b^{-1}c^{-1}b^2.$$

In the indicated square of the latter, we replace b^2cb^{-2} by $c^{-1}b^2c^{-1}b^2c^{-1}$, in view of (5_4), and transform by cb^2, and get

$$c^{-1} \cdot bc^{-1}b^{-1}c \cdot b^2c^{-1}b^2 \cdot cbc^{-1}b^{-1} \cdot c^{-1}b^2cb^2$$
$$= c^{-1} \cdot c^{-1}bcb^{-1} \cdot b^2c^{-1}b^2 \cdot bcb^{-1}c^{-1} \cdot c^{-1}b^2cb^2 \qquad [\text{by } (5_3)]$$
$$= cb \cdot cbc^{-1}b^{-1} \cdot cb^{-1}cb^2cb^2$$
$$= cb \cdot bcb^{-1}c^{-1} \cdot cb^{-1}cb^2cb^2 = (cb^2)^4 = I \qquad [\text{by } (5_4)].$$

To verify (3_2), we transform by b^2c^{-1} and get

$$cb^2cb^2 \cdot cb^{-1}cb \cdot cb^2cb^2 \cdot cb^{-1}cb$$
$$= b^2c^{-1}b^2c^{-1} \cdot cb^{-1}cb \;\; b^2c^{-1}b^2c^{-1} \cdot cb^{-1}cb$$
$$= b^{-1}(b^{-1}c^{-1}bc)^2b = I.$$

To verify (2_1), we note that

$$ab^{-1}ab = cb^{-1}cb \cdot cb^{-1}cb^{-1}cbcb$$
$$= cb^{-1}cb(cb^{-1}cb^{-1}cbcb)^{-1} = cb^{-2}c^{-1}bc^{-1}bc^{-1}.$$

Cubing the inverse and transforming by c, we get (8).

To verify (2_2), we note that

$$ab^{-2}ab^2 = ac^{-1} \cdot cb^{-2}ab^2 = ac^{-1} \cdot b^{-2}a^{-1}b^2c^{-1}$$
$$= cb^{-1}cb^{-1}c^{-1}b^{-1}c^{-1}bc^{-1}b^2c^{-1}.$$

Transforming its square by cb^2, we get

$$bcb^{-1}c^{-1}\cdot b^{-1}c^{-1}bc^{-1}\cdot bcb^{-1}c^{-1}\cdot b^{-1}c^{-1}bc^{-1}$$
$$= cbc^{-1}b^{-1}\cdot b^{-1}c^{-1}bc^{-1}\cdot cbc^{-1}b^{-1}\cdot b^{-1}c^{-1}bc^{-1}$$
$$= cbc^{-1}b^{-2}c^{-1}b^{2}c^{-1}b^{-2}c^{-1}bc^{-1}.$$

Transforming by cb and taking the inverse, we get $(b^2c)^4 = I$.

2. In a paper entitled "The abstract group simply isomorphic with the group of linear fractional transformations in a Galois field," communicated November 2, 1902 to the London Mathematical Society, the writer shows that the group G is generated by three operators subject to the relations

$$(9) \qquad T^2 = I, \quad S_1^3 = I, \quad S_j^3 = I, \quad S_1S_j = S_jS_1,$$

$$(10) \qquad (S_1T)^3 = I, \quad (S_jT)^4 = I, \quad (S_jS_1TS_j^{-1}S_1T)^2 = I.$$

From these we obtain relations (1), (2), (3), if we set

$$a = T, \quad b = S_jT, \quad c = S_1.$$

This is evident for relations (1). Also,

$$(ab^{-1}ab)^3 = (S_j^{-1}TS_jT)^3 = S_j^{-1}\cdot TS_jTS_j\ S_jTS_jT\cdot S_j^{-1}TS_jT$$
$$= S_j^{-1}\cdot S_j^{-1}TS_j^{-1}T\cdot TS_j^{-1}TS_j^{-1}T\cdot S_j^{-1}TS_jT = (S_jT)^4 = I.$$

Also (2_2) and (3_2) follow from (9) and $(S_jT)^4 = I$, while (3_1) follows from (9) and the first and third relations (10). We thus obtain a new proof that (9) and (10) define G.

We may readily derive directly from (9) and (10) a complete set of relations between the two generators $b = S_jT$ and $c = S_1$ of G. We note that, from (10_1),

$$T = S_1TS_1TS_1 = S_1\cdot TS_j^{-1}\cdot S_1\cdot S_jTS_1 = cb^{-1}cbc.$$

We therefore have

$$S_1 = c, \quad T = cb^{-1}cbc, \quad S_j = bc^{-1}b^{-1}c^{-1}bc^{-1}.$$

Then $(S_jT)^4 = I$ follows from (5_1), $S_1^3 = I$ from (5_2), $T^2 = I$ from (5_3), $S_1S_j = S_jS_1$ from (5_3). Thus

$$S_1S_j = cbc^{-1}b^{-1}\cdot c^{-1}bc^{-1} = bcb^{-1}c^{-1}\cdot c^{-1}bc^{-1}$$
$$= bc\cdot b^{-1}cbc^{-1} = bc\cdot cb^{-1}c^{-1}b = S_jS_1.$$

Since $S_1 T = c^{-1}b^{-1}cbc$ is the transform of c by bc, it is of period three.

The final relation (10) becomes

$$(bc^{-1}b^{-1}c \cdot b^{-1}cbc)^2 = (c^{-1}bcb^{-1} \cdot b^{-1}cbc)^2 = (c^{-1}bcb^2cbc)^2$$
$$= c^{-1}b(cb^2)^4b^{-1}c = I.$$

Since S_j is commutative with S_1, the condition $S_j^3 = I$ follows from $(b^{-1}c^{-1}b^2c^{-1})^3 = I$ or $(cb^2cb)^3 = I$.

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NOTE ON A PROPERTY OF THE CONIC SECTIONS.

BY PROFESSOR H. F. BLICHFELDT.

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IT is easily proved that if P, Q, R are any three points on the conic $Ax^2 + By^2 = 1$, and O the center of the conic, then the areas of the triangles OPQ, OPR, OQR will satisfy an equation independent of the position of the points P, Q, R. If a, b, c are the areas in question, this equation is

$$(1) \quad a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 + 16ABa^2b^2c^2 = 0.$$

Now we can prove that such an invariant relation is possible for no plane curves except the central conics; *i. e., if we seek a plane curve C and a point O in its plane such that, if P, Q, R are any three points on C, the triangles OQR, ORP, OPQ are connected by a relation independent of the coördinates of the points P, Q, R, we find C to be a central conic section and O its center.*

To prove this theorem, let O be the origin of coördinates, and let the coördinates of P, Q, R be respectively x_1, y_1; x_2, y_2; x_3, y_3. Then twice the areas of the three triangles are

$$2a = \pm (y_2x_3 - y_3x_2), \quad 2b = \pm (y_3x_1 - y_1x_3),$$
$$2c = \pm (y_1x_2 - y_2x_1),$$