for further work whether in physics or mathematics or theoretical mechanics by beginning with a general course which includes the elements of many branches of mechanics than by beginning with a thorough course in any one branch.

Professor Cellérier's Cours de Mécanique was in many ways an interesting and most valuable book when it appeared. We are inclined even now to recommend it to all who wish to follow the method in general use among engineers—the method of starting mechanics with a discussion of statics.

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KIEPERT'S CALCULUS.

Grundriss der Differential- und Integral-Rechnung. Von Dr. LUDWIG KIEPERT. I. Theil: *Differential-Rechnung.* 9te Auflage. II. Theil. *Integral-Rechnung.* 7te Auflage. Hannover, Helwingsche Verlagsbuchhandlung, 1900–1901.

A WORK which has gone through so many editions in a country where so much attention is given to mathematical instruction should have its merits; nor in this case will the reader fail to discover them. These are clearness of statement, carefully drawn illustrations in great number, minute explanations, warnings against probable mistakes on the part of the student. Nevertheless from the very fact that we have a ninth reproduction of an older work there is a certain clinging to tradition which impairs the usefulness of the book.

For example, it gives the student a better idea both of the theory and the range of application, if the differential and integral calculus be treated simultaneously; moreover, he begins at once to know something of integration, a matter of great practical importance. Again, a subject so rich in applications to the various sciences and industrial arts should be presented along with these applications. Thus can the subject be made alive to the student rather than a dead and tiresome exercise in formula grinding. It may be, however, that a problem book, used in conjunction with the text, will supply this deficiency, and of course every good teacher will add problems of his own.

The first volume has an introduction concerning functions, limits, infinitesimals of various orders, continuity, the binomial theorem for positive integral exponents, geometrical progression and the number e. It would seem as if the notion of inverse function, always difficult for a student to grasp, required rather more illustration than is given.

Under the head of infinitesimals the derivative is introduced and some illustrations, involving but slight calculation, given from geometry and mechanics. Further such illustrations might well be given, and some of them might precede the definition and the notation. As a rule, notation ought to be brought in only when necessary to express a previously acquired idea.

Continuity and discontinuity are illustrated graphically and then the same cases that gave the graphs are treated algebraically. The chapter closes with a simple proof that e is no rational number.

The calculus proper opens with the differentiation of the algebraic, the exponential and the circular functions. The double sign for the derivatives of the inverse circular functions might be more fully explained, and in introducing the notation

$$dy = df(x) = f'(x)dx,$$

there should be insistence on the fact that the equality has reference to the ratio unity of the two members, as Δx and Δy vanish to dx and dy.

The hyperbolic functions are given a chapter; but it is not noted how by defining circular and hyperbolic functions as the ratios of areas a complete analogy is established between them. (See BULLETIN, Volume 1, page 155.)

The introduction of derivatives of higher orders leads at once to the Taylor-Maclaurin theorem, to which with applications are devoted sixty pages. For careful step by step development this chapter can hardly be surpassed. The theorem is first given for rational algebraic functions, as, for example, the binomial expansion for a positive integral exponent. It is next shown that in certain particular cases, as when the development leads to a decreasing geometric series, R_n, the remainder after n terms, vanishes. The mean-value theorem is then proved and illustrated so that the student is ready for the usual determination of R_n. The importance of the determination is illustrated by considering the increasing geometric series. The usual applications to expansion of functions are given; yet little use is made of various auxiliary methods such as multiplication and division, integration and differentiation of known expansions. This would, to be sure, anticipate matters to be brought up in the chapter on series. But certainly they could somewhere be well given, and possibly, in an elementary treatise, would be of more value than the rather difficult discussion of the conditions under which the expansion of a binomial whose two terms are numerically equal is valid. A graphic illustration, such as Klein's, of the way in which a function though converging within certain limits diverges without those limits would be an aid to clearness. (See Lamb's Calculus, page 572.)

There follows a good chapter on series. But the treatment is purely algebraic and one misses such geometric illustrations as those in Professor Osgood's pamphlet. The convergence of series of sines and cosines is touched upon. Here it would have been interesting to have made a statement about series that are convergent while the series gotten by differentiating them term by term are not.

The chapter on maxima and minima is remarkable for the great variety of examples worked out. The case where neither x nor y is an implicit function of the other is reserved for a later chapter.

In the treatment of indeterminate forms the usual proof that when $f(x) = \infty$ and $\varphi(x) = \infty$

$$A \equiv \frac{f(x)}{\varphi(x)} = \frac{f'(x)}{\varphi'(x)}$$

is completed by taking account of the possibility that $A = 0$. There is no suggestion that there are other than the usual indeterminate forms; as for example, $\log_1 1$, $\log_\infty \infty$, $\log_0 0$, $\log_\infty 0$, $\log_0 \infty$. "Indeterminate" meaning "yet to be determined" should be distinguished from real indetermination as dy/dx at the origin on the logarithmic spiral

$$\sqrt{x^2 + y^2} = e^{\tan^{-1}(y/x)}.$$

In the chapter on differentiation of implicit functions help would be afforded by a geometrical illustration of the formula

$$\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}.$$

Under interchange of variables, while the simplification of the expression

$$\frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\frac{dx^3}{dt^3}}$$

when $x = t$ or when $y = t$ is noted, it is not remarked that the simplification is secured by supposing x or y merely linear in t. The notions of dependent and independent variation need careful definition.

The next two chapters give the usual geometrical applications to tangents, curvature, evolutes, and so on. In the applications to polar coördinates the conception of r as a revolving y-axis and the perpendicular thereto through the origin as a revolving x-axis would enable the student to easily see just how and why the cartesian formulæ pass over into the polar by merely writing dr and rdt in place of dy and dx.

Algebraic investigations on complex numbers, on the roots of equations including methods of approximation thereto, and on determinants occupy the most of 120 pages. The space given to Graefe's method of approximation, since the method requires special conditions for its successful application and inasmuch too as no criterion is furnished for the error in the calculation, might possibly better be occupied with a demonstration that every equation has a root.

The concluding section of the differential calculus deals with functions of several variables. Only the simpler formulæ are derived and the simpler geometrical applications made. Among these are the theory of envelopes of systems of plane curves and the theory of multiple points of plane curves. For the latter the general formula is derived giving the slopes of all the branches through the point in question. Thus, for a triple point, we have

$$\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}\right)^{(3)} \equiv F_{111} + 3F_{112}\frac{dy}{dx} + 3F_{122}\frac{dy^2}{dx^2} + F_{222}\frac{dy^3}{dx^3} = 0,$$

where the subscripts 1 and 2 denote differentiation with regard to x and y respectively.

The Taylor-Maclaurin theorem is extended to functions of several variables and there is a brief treatment of homogeneous functions.

The concluding chapter of the section contains a fairly complete treatment of maxima and minima for functions of several variables.

It is gratifying to find the student warned against supposing that, if two functions of two or more variables vanish with those variables, the function of the higher degree is zero of a higher order. One would suppose the method of Newton's parallelogram old enough so that even English and American texts should not be vitiated by such a conclusion. (See BULLETIN, volume 4, page 535.)

For n variables the conditions for maxima and minima are stated as follows:

$$u = f(x_1, x_2, \cdots, x_n)$$

is a minimum if the first partial derivatives $f_1(x_1, x_2, \cdots x_n)$, $f_2(x_1, x_2, \ldots x_n)$, $\cdots$, $f_n(x_1, x_2, \cdots x_n)$ *all vanish, and if*

$$D_1 > 0,\ D_2 > 0,\ D_3 > 0, \cdots, D_n > 0.$$

Here

$$D_\alpha = \begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1\alpha} \\ f_{21} & f_{22} & \cdots & f_{2\alpha} \\ \cdots & \cdots & \cdots & \cdots \\ f_{\alpha 1} & f_{\alpha 2} & \cdots & f_{\alpha\alpha} \end{vmatrix}$$

for $\alpha = 1, 2, \cdots, n$

On the other hand, u is a maximum if the first partial derivatives vanish as before, and if the determinants D_α *with even index are all positive while those with odd index are all negative.*

The case of maxima or mimima when some of the first partial derivatives become infinite is not considered. Implicit functions are given a special treatment.

The volume contains tables for the Gudermannian relation and for hyperbolic functions, together with a table of 233 important formulæ—practically all those derived or referred to in the text. It is illustrated by 171 carefully drawn diagrams.

The volume on the integral calculus begins with a somewhat detailed explanation of the nature and geometric meaning of an integral. The usual representation of an integral by an area is given so that the derivative is represented by an ordinate. Since in the differential calculus the derivative has been represented by a slope so that the primitive was an ordinate, there should perhaps be greater emphasis laid upon the constancy of the relation in spite of the change in representation.

A demonstration is given, first graphically and then by analysis, that every continuous function has an integral. It is stated, but not proved, nor is an illustration given, that not every continuous function has a derivative.

The various simpler cases of integration are gone over rather rapidly. The author objects to the term "integration by parts," proposing instead "partial integration." But there seems hardly sufficient reason for changing a usage so well established, and besides there is even a better application for the term proposed by him, viz., the integration of results gotten by partial successive differentiation. Thus the value of

$$\int\int f(xy)\,dxdy$$

when the limits for y are variable is gotten by first integrating with regard to y as if x were constant. (See Greenhill's Calculus, page 283.)

The applications to quadrature, cubature, rectification and complanation are enriched by many examples.

After the geometric excursus the author returns to the matter of breaking up a fraction into partial fractions giving the demonstration of that possibility with satisfactory thoroughness. In determining the numerators the rather tedious method of undetermined coefficients is exclusively employed, no mention being made of Hermite's division method. (See Greenhill's Calculus, page 387.) If the reader will compare the labor required by the two methods on such a case as breaking up

$$\frac{1}{(x^2-1)^3}$$

into partial fractions he will have no doubt of the great superiority of the less used of the two.

Integration over discontinuities is briefly but instructively treated. A good example is

$$\int_{-a}^{+b}\frac{dx}{x}$$

This is

$$\lim_{\gamma=0}\int_{-a}^{-\gamma}\frac{dx}{x}+\lim_{\delta=0}\int_{+\delta}^{+b}\frac{dx}{x}$$

$$=\lim_{\gamma=0} ln\left(\frac{\gamma}{a}\right)+\lim_{\delta=0} ln\left(\frac{b}{\delta}\right)$$

$$=ln\left(\frac{b}{a}\right)+\lim\cdot ln\cdot\left(\frac{\gamma}{\delta}\right)_{(\gamma=\delta)}=ln\left(\frac{b}{a}\right)$$

$=$ *Cauchy's principal value.*

The integration of series term by term is made to lead up to the calculation of the elliptic integrals of the first and second kind. Though it is remarked that these integrals have many important applications, none whatever are made.

A number of determinations of definite integrals are given as well as approximations by Simpson's rule and by Gaussian quadrature, the errors being estimated. The more accurate rule of Weddle is not mentioned. A good chapter on planimeters and integraphs would be interesting in this connection.

The error into which one may sometimes fall on substituting y for a function of x under an integral when dy/dx changes its sign within the limits of integration is clearly pointed out by means of the example

$$\int_1^7 (x^2 - 6x + 13)\,dx = 48.$$

If here we make the substitution $y = x^2 - 6x + 13$ we apparently get

$$\pm \tfrac{1}{2}\int_8^{20} \frac{y\,dy}{\sqrt{y-4}} = \pm \tfrac{80}{3},$$

an absurdity. Closer examination shows that the minus sign should be used while x varies from 1 to 3 and y from 8 to 4, while the plus sign is right when x varies from 3 to 7 and y from 4 to 20.

The work concludes with an elementary treatment of differential equations, excellent from the algebraic point of view. The proof that every differential equation has an integral is given with great attention to detail. Even if the student is unable to read this, he will at least get an idea of the extreme care necessary if rigorous demonstration is to be had. Difficulties cannot be safely avoided even in the elements; for easy and plausible proofs are liable to become dangerous poisons to the mind that imbibes them. None but geometrical applications are given and these naturally quite fail us when we come to equations of the nth order.

Like the differential calculus, the integral has at the end a collection of all the formulas used. Twelve pages of these formulas deal entirely with differential equations, and if classified under appropriate headings would furnish a good working key.

There should be a table of gamma functions to go with the formulas for definite integrals, and it would seem that if the elliptic integrals are introduced at all there should be tables of E and F, the complete integrals. An index is needed.

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