we shall have the definition given by H. Weber, *loc. cit.* That these postulates 1, 2, 3′, 4′, 5*a* are mutually independent (when $n > 2$) has already been shown in the writer's previous paper (page 300).

It should be noticed, however, that postulates 1, 2, 3′, 4′, 5*b* would not be sufficient to define an *infinite* group, since the system of positive integers, with $a \circ b = a + b$, satisfies them all, and is not a group.

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DETERMINATION OF ALL THE GROUPS OF ORDER p^m, p BEING ANY PRIME, WHICH CONTAIN THE ABELIAN GROUP OF ORDER p^{m-1} AND OF TYPE (1, 1, 1, ⋯).

BY PROFESSOR G. A. MILLER.

(Read before the San Francisco Section of the American Mathematical Society, May 3, 1902.)

LET $t_1, t_2, \cdots, t_{m-1}$ represent a set of independent generators of the abelian group H of type (1, 1, 1, ⋯). It is well known that the order of the group of isomorphisms ϑ of H is $p^{\frac{(m-1)(m-2)}{2}}(p-1)(p^2-1)\cdots(p^{m-1}-1)$. One of its subgroups ϑ_1 of order $p^{\frac{(m-1)(m-2)}{2}}$ is composed of all the operators of ϑ which correspond to the holomorphisms of H in which t_α $(\alpha = 2, 3, \cdots, m-1)$ corresponds to itself multiplied by some operator in the group generated by $t_1, t_2, \cdots, t_{\alpha-1}$. The number of conjugates of ϑ_1 under ϑ is clearly equal to the order of ϑ divided by $p^{\frac{(m-1)(m-2)}{2}}(p-1)^{m-1}$. We shall first determine the number of sets of subgroups of ϑ_1 which are conjugate under ϑ. It may be observed that even characteristic subgroups of ϑ_1 may be conjugate under ϑ. For instance, the octic group has a characteristic subgroup of order two and four other subgroups of this order, yet all of these subgroups are conjugate under ϑ when the latter is the simple group of order 168.

All the holomorphisms of H may be obtained by establishing isomorphisms between H and its subgroups and letting the product of two corresponding operators in these isomorphisms correspond to the original operator of H.*

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* BULLETIN, vol. 6 (1900), p. 337.

If two operators of ϑ_1 correspond to such holomorphisms in which the subgroups are of different orders, they evidently belong to different sets of conjugates under ϑ. Hence $m - 2$ sets of conjugate subgroups of ϑ_1 can be obtained in this manner. It remains to determine the number of sets of conjugates when the multiplying subgroups are of the same order h_1. When h_1 is either p or p^{m-2} it is not difficult to see that the corresponding operators of ϑ_1 are conjugate under ϑ.

In general, suppose that in any holomorphism of H, which corresponds to an operator in ϑ whose order is a power of p, the above set of independent generators of H has been so selected that those which correspond to themselves come first; and let t_α' represent any operator of the group generated by $t_1, t_2, \cdots, t_\alpha$ which is not in its subgroup generated by $t_1, t_2, \cdots, t_{\alpha-1}$. The operator of ϑ which corresponds to this holomorphism transforms some t_α', which is not found in the multiplying subgroup, into $t_{\alpha_1}t_\alpha'$, t_{α_1} into $t_{\alpha_2}t_{\alpha_1}, \cdots, t_{\alpha_{n-1}}$ into $t_{\alpha_n}t_{\alpha_{n-1}}$, and t_{α_n} into itself. If $t_{\alpha_1}, t_{\alpha_2}, \cdots, t_{\alpha_n}$ do not generate the multiplying subgroup, *i. e.*, if h_1 exceeds p^n, we may find another series of transforms in the same manner and thus arrive at another number n_1. The necessary and sufficient condition that two holomorphisms may correspond to conjugate operators under ϑ is that the numbers $n, n_1, \cdots$ and $n', n_1', \cdots$ which correspond to these holomorphisms are the same.

For instance, when $m = 5$, ϑ_1 contains four sets of conjugate cyclic subgroups under ϑ. Two of these correspond to the case where the multiplying subgroup is of order p^2 and the other two correspond to the cases where this order is p or p^3. When $m = 6$, the number of these conjugate sets is clearly six. In general, *when $h_1 = p^\lambda$ the number of sets of cyclic subgroups of ϑ_1, such that each set includes one complete system of conjugates under ϑ, is equal to the number of partitions of λ with respect to addition when the number of addends does not exceed $m - 1 - \lambda$.* Hence, it follows from Euler's theorem: the number of partitions of $n + r$ into r parts is equal to the number of ways in which n can be represented as the sum of one or more of the r numbers $1, 2, \cdots, r$ (with repetitions), that the total number of such sets of cyclic subgroups in ϑ_1 is equal to the number of partitions of $m - 1$ into at least two parts with respect to addition. This number does not include the identity.

The orders of the operators of ϑ_1 can be directly obtained by means of the formula*

* BULLETIN, vol. 7 (1901), p. 351.

$$t^{-n}s_\alpha t^n = s_{\alpha+n}s^n_{\alpha+n-1}\cdots s_{\alpha+n-r}^{n(n-1)\cdots(n-r+1)/r!}\cdots s^n_{\alpha+1}s_\alpha$$

whenever $t^{-1}s_\beta t = s_{\beta+1}s_\beta$ ($\beta = \alpha, \alpha+1, \cdots, \alpha+n-1$). If i_1 represents any operator of ϑ_1 which corresponds to the numbers $n, n_1, \cdots$ its order is p^{α_1}, where α_1 is the smallest power of p which exceeds each of the numbers $n, n_1, \cdots$. When $\alpha_1 > 1$ the conjugate set to which i_1 belongs does not give rise to any group of order p^m. When $n = 1$ and all the other numbers $n_1, \cdots$ are zero, the conjugate set to which i_1 belongs gives rise to three groups whenever $p > 2$ and $m > 3$.* One of these is conformal with the abelian group of type (1, 1, 1, $\cdots$) while the other two are conformal with the abelian group of type (2, 1, 1, $\cdots$). In one of the last two groups each of the p^{m-2} cyclic subgroups of order p^2 is invariant while none of these subgroups is invariant in the other.

When p is odd and $m = 3$ there are just two groups, which are conformal respectively with the abelian groups of types (1, 1, 1, $\cdots$) and (2, 1, 1, $\cdots$). Each of the p^{m-2} cyclic subgroups of order p^2 in the latter group is invariant. When $p = 2$ and $m = 3$ there is only one group, viz., the octic group; when $m > 3$ there are two groups, one being conformal with the abelian group of type (2, 1, 1, $\cdots$) while the other contains $2^{m-1} + 2^{m-2} - 1$ operators of order 2, the remaining operators being of order 4. Each of the groups which have been considered contains $p + 1$ abelian subgroups of order p^{m-1} while remaining groups contain no abelian subgroups of this order except H.

In general, if p exceeds all the numbers $n + 1, n_1 + 1, \cdots$, then i_1 and H generate a group which contains only operators of order p besides identity. For, if s is any operator of H, then

$$(st)^p = st\,st\,st \cdots = st\,st^{-1}\,t^2\,st^{-2}\,t^3 \cdots t^{p-1}\,st = 1.$$

For each of the numbers $n, n_1, \cdots$ which are distinct there is clearly one additional group in which H is transformed in the same manner as i_1 transforms it. If the sum of the numbers $n + 1, n_1 + 1, \cdots$ is less than $m - 1$; *i. e.*, if H contains invariant operators which are not commutators, there is one additional group in which H is transformed according to i_1. All of these groups are conformal with the abelian group of type (2, 1, 1, 1,$\cdots$).

When p is equal to one or more of the numbers $n + 1$,

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*Only non-abelian groups of order p^m are considered in this paper, since the two abelian groups which contain H are well known.

$n_1 + 1, \cdots$, then the group generated by i_1 and H contains operators of order p^2 and the remarks in regard to additional groups apply only to the remaining numbers and to the invariant operators of H which are not commutators. As i_1 and its conjugates cannot give rise to any group of order p^m when p is less than some one of the numbers $n + 1$, $n_1 + 1, \cdots$, all the groups of this order which contain H can be readily obtained by the above considerations. It may be observed that this includes all the groups of order p^m in which every operator is of order p whenever $m < 5$, since every group of order p^4 contains an abelian subgroup of order p^3.

STANFORD UNIVERSITY,
April, 1902.

A CLASS OF SIMPLY TRANSITIVE LINEAR GROUPS.

BY PROFESSOR L. E. DICKSON.

1. IN the study of the group defined for any given field by the multiplication table of any given finite group,* it is necessary to discuss the types of simply transitive linear homogeneous groups G whose transformations can be given the form

$$(1)\qquad \begin{gathered} \xi_1' = \eta_1\xi_1, \quad \xi_2' = \eta_2\xi_1 + \eta_1\xi_2, \quad \xi_3' = \eta_3\xi_1 + \alpha\xi_2 + \eta_1\xi_3, \\ \xi_4' = \eta_4\xi_1 + \beta\xi_2 + \gamma\xi_3 + \eta_1\xi_4, \\ \xi_5' = \eta_5\xi_1 + \lambda\xi_2 + \mu\xi_3 + \nu\xi_4 + \eta_1\xi_5, \cdots. \end{gathered}$$

Here $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \cdots$ are the independent parameters, while $\alpha, \beta, \gamma, \lambda, \cdots$ are linear homogeneous functions of the η_i. Burnside † was led to the erroneous conclusion that every such group G is an abelian group. He first concludes that the expression for ξ_t' contains only the parameters $\eta_1, \cdots, \eta_t$ and contains η_t only in the first term $\eta_t\xi_1$. That this result need not be true is shown by a consideration of the simply transitive group of quaternary transformations

$$(2)\qquad \begin{gathered} \xi_1' = \eta_1\xi_1, \quad \xi_2' = \eta_2\xi_1 + \eta_1\xi_2, \quad \xi_3' = \eta_3\xi_1 + \alpha\xi_2 + \eta_1\xi_3, \\ \xi_4' = \eta_4\xi_1 - \frac{a_3}{a_4}\alpha\xi_2 + \eta_1\xi_4, \end{gathered}$$

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* For the case of a continuous field, Burnside, *Proc. Lond. Math. Soc.*, vol. 29 (1898), pp. 207–224, 546–565; for an arbitrary field, Dickson, *Transactions*, vol. 3 (1902), pp. 285–301.

† *Proc. Lond. Math. Soc.*, vol. 29, pp. 552–553.