absolute invariants. The binary linear forms give rise in this way to the cross ratio group discussed by Professor Moore in the *American Journal of Mathematics*, 1900. The groups discussed in detail are those relating to any set (n) of linear forms in any number (k) of variables for both the associated system and the system of absolute invariants, the number of variables being

$$\binom{n}{k} - \binom{n-k}{k} \quad \text{and} \quad (k-1)(n-k-1)$$

respectively. The process considered is capable of extension in several directions, for example to forms which are not all of the same order, and to the formation of infinite groups.

Mr. Risteen's paper is in abstract as follows: If Euclid's parallel axiom is true, then it is known that the parallax of a double star can be obtained in either of two ways, namely, 1° by the usual micrometric measures, and 2° by observing the relative velocities of approach or recession of the component stars in the line of sight. If the parallel axiom is not true, and space is admitted to be hyperbolic, then the known trigonometric relations that hold for hyperbolic space, and which were given by Lobachevsky, enable us to combine the micrometric measures and the spectroscopic ones, so as to obtain a single estimate of the star's distance, together with the value of the "constant of space" that occurs in Lobachevsky's equations, but whose value has not yet been determined. The same principle applies equally well if space is elliptic. Lobachevsky's method of finding a limiting value for his constant is unsound. A numerical value of the "constant of space" can hardly be found at present, because the necessary spectroscopic data cannot yet be had.

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THE INFINITESIMAL GENERATORS OF PARAMETER GROUPS.

BY PROFESSOR T. J. I'A. BROMWICH.

(Read before the American Mathematical Society, April 26, 1902.)

§ 1. Dr. Slocum has given (Bulletin, January, 1902, page 156) a method for calculating the infinitesimal generators of the parameter group which belongs to a group of known structure. On reading his paper it seemed to me that certain results of my own * might be applied with advantage ; and on examination, I found that my process was less laborious than Slocum's and, I believe, more accurate. For I have found certain differences in our results, and, so far as I can test them, it seems that Slocum's results are less satisfactory than mine, in cases where the two differ.

§ 2. Slocum proves (l. c., page 158, equation (3)) that

$$\beta = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n!} \gamma_n$$

where

$$\gamma_n = (\alpha, \gamma_{n-1}), \qquad (n = 2, 3, 4, \cdots)$$

and

$$\gamma_1 = \gamma,$$

the parenthesis denoting Lie's alternant (or Klammerausdruck). Here α, β, γ are three differential operators of the given group ; and Slocum shows that the determination of the differential operators of the parameter group is equivalent to finding the parameters of γ when those of β are given.

I shall now explain how to solve the last problem by applying certain theorems on matrices. To make the application clear, consider first the alternant (α, λ) where

$$\alpha = \sum_{k=1}^{r} a_k X_k, \quad \lambda = \sum_{k=1}^{r} l_k X_k$$

and the a's and l's are arbitrary parameters. If the structure of the given group is defined by

$$(X_p, X_q) = X_p X_q - X_q X_p = \sum_{s=1}^{r} c_{pqs} X_s \quad (p, q = 1, 2, \cdots, r),$$

we shall have

$$(\alpha, \lambda) = \Sigma_{p,q}\, a_p l_q (X_p, X_q) \quad (p, q = 1, 2, \cdots, r),$$

$$= \Sigma_{p,q}\, a_p l_q \left(\sum_{s=1}^{r} c_{pqs} X_s \right),$$

$$= \sum_{q=1}^{r} \sum_{s=1}^{r} \alpha_{qs} l_q X_s,$$

where

* I should, perhaps, remark that these results were worked out in October, 1900, but were not published, owing to the advice of friends, who wished me to elaborate some further details. Unfortunately, I have not yet been able to carry this out In justice to Dr. Slocum, I ought to add that I had not then used my method for parameter groups.

$$a_{qs} = \sum_{p=1}^{r} a_p c_{pqs},$$

and so a_{qs} depends only on the a's and the structure of the given group. Hence the parameters of (a, λ) are derived from those of λ by means of a linear substitution; and this linear substitution has a matrix A, whose elements are a_{qs}.

In particular the parameters of γ_n are derived from those of γ_{n-1} by means of the substitution A; and hence (according to the definition of the square of a matrix) those of γ_n are derived from those of γ_{n-2} by means of A^2; and so on. Thus, finally, since $\gamma_1 = \gamma$, it follows that γ_n is derived from γ by means of a linear substitution whose matrix is A^{n-1}.

Turning now to the equation for β, it follows that the parameters of β are derived from those of γ by means of the substitution associated with the matrix

$$E - \frac{1}{2!}A + \frac{1}{3!}A^2 - \frac{1}{4!}A^3 + \cdots = \frac{[E - \exp.(-A)]}{A}$$

$$= B, \text{ say},$$

where E is the unit (or identical) matrix. Hence, to find the parameters of γ from those of β, we shall need the matrix B^{-1}, which is reciprocal to B. In terms of A, we have

$$B^{-1} = A/[E - \exp.(-A)] = f(A) \text{ say}.$$

Now I come to the essential novelty of my method; in order to find B^{-1}, Slocum calculates B, and forms its reciprocal; but I shall show that *to find $B^{-1} = f(A)$, we need not find B first, but can find $f(A)$ directly from the properties of A and the form of the function $f(t)$, that is,*

$$f(t) = t/(1 - e^{-t}).$$

For, according to Frobenius and Stickelberger,* *any* function of a matrix (provided that such a function has a meaning) can be found as follows:

Take the matrix $(tE - A)$, where t is arbitrary, and form the reciprocal† matrix $(tE - A)^{-1}$; let the reciprocal be

* For references see § 2 of a paper by the present writer, *Proc. Camb. Phil. Soc.*, vol. 11 (1900), p. 75. The rule is there compared with the equivalent, but less convenient, rules found by Buchheim and Taber.

† If a_{qs} is an element of a matrix, the corresponding element of the reciprocal matrix is the minor of a_{sq} (with proper sign), divided by the determinant of the original matrix $|a_{sq}|$. In most of the cases considered here it is easier to construct the reciprocal by using the property that the "product" of a matrix by its reciprocal is E (*i. e.*, the unit matrix). It is perfectly easy to show that the two methods are equivalent.

arranged in a series of partial fractions, corresponding to the various factors of the determinant $|tE - A|$. Thus, suppose that, corresponding to a factor $(t - t_0)$, we get the terms in $(tE - A)^{-1}$

$$\frac{T_m}{(t-t_0)^m} + \frac{T_{m-1}}{(t-t_0)^{m-1}} + \cdots + \frac{T_1}{t-t_0},$$

where T_m, T_{m-1}, $\cdots$, T_1 are certain matrices (independent of t), and $(t - t_0)^m$ is an invariant factor of the determinant. Then we have, for any function

$$f(A) = \Sigma \left[f(t_0) T_1 + f'(t_0) T_2 + \cdots + \frac{1}{(m-1)!} f^{m-1}(t_0) T_m \right],$$

the summation being extended to all *distinct* factors of $|tE - A|$.

We shall apply this rule to find $B^{-1} = f(A)$.

One great advantage in this process is that we can separate the complications due to the form of $f(t)$ from those due to the form of A. Thus we should first calculate $f(t_0)$, $f'(t_0)$, $f''(t_0)$, $\cdots$, which will be available for *all* the groups; and the fresh calculations in each type relate only to the determination of T_1, T_2, T_3, $\cdots$. In practice, I find it less cumbrous to split up each element of the matrix $(tE - A)^{-1}$ into its partial fractions, rather than to write out the matrices T_1, T_2, T_3, $\cdots$ at length; then, wherever $1/(t - t_0)^{r+1}$ occurs, we have only to write $\frac{1}{r!} f^r(t_0)$ in its place, in order to deduce $f(A)$ from $(tE - A)^{-1}$.

For the special form of $f(t)$, which is required for the present purpose, we have, if $p = \frac{t_0}{1 - e^{-t_0}}$,

$$f(t_0) = p, \quad f'(t_0) = p\left(1 + \frac{1-p}{t_0}\right),$$

$$\frac{1}{2!} f''(t_0) = \frac{p^2(p-1)}{t_0^2} - \frac{3p^2}{2t_0} + \frac{p}{t_0} + \frac{p}{2}$$

and in the special case $t_0 = 0$, we have

$$f(0) = 1, \quad f'(0) = \tfrac{1}{2}, \quad \tfrac{1}{2} f''(0) = \tfrac{1}{12}, \quad \tfrac{1}{6} f'''(0) = 0,$$

where the continuation depends on Bernoulli's numbers. The coefficients given above are all that will be needed in the groups of *less* than five parameters. To include the case of five parameters, we should want one more of each set; and so on.

Finally, when the matrix $f(A)$ is determined, it follows from Slocum's equation (7) that the kth operator of the parameter group has the coefficients found in the kth column of $f(A)$.

§ 3. To illustrate the method, I first took Type I of the three-parameter groups (page 161, l. c.). Here

$$(X_1, X_2) = X_1, \quad (X_1, X_3) = 2X_2, \quad (X_2, X_3) = X_3.$$

This gives for the matrix A

$$\begin{bmatrix} -\ a_2 & +a_1 & 0 \\ -2a_3 & 0 & +2a_1 \\ 0 & -a_3 & +\ a_2 \end{bmatrix}$$

and so

$$tE - A = \begin{bmatrix} t + \ a_2 & -a_1 & 0 \\ +2a_3 & t & -2a_1 \\ 0 & +a_3 & t - a_2 \end{bmatrix}.$$

Hence, forming the reciprocal, we find

$$(tE - A)^{-1} = \frac{1}{t(t^2 - \varphi^2)}$$

$$\times \begin{bmatrix} t(t - a_2) + 2a_1a_3 & a_1(t - a_2) & 2a_1^2 \\ -2a_3(t - a_2) & t^2 - a_2^2 & 2a_1(t + a_2) \\ 2a_3^2 & -a_3(t + a_2) & t(t + a_2) + 2a_1a_3 \end{bmatrix}$$

where $\varphi^2 = a_2^2 - 4a_1a_3$, as used by Slocum; and it should be remarked that *every* element is divided by the denominator $t(t^2 - \varphi^2)$.

According to the rule we have now to divide each element into partial fractions, and then replace

$1/t$ by 1, $1/(t - \varphi)$ by $\varphi/(1 - e^{-\phi})$, $1/(t + \varphi)$ by $\varphi/(e^{\phi} - 1)$,

respectively, in order to obtain $f(A)$ from $(tE - A)^{-1}$.

It will save trouble, however, to put down the partial fractions for

$$\frac{1}{t(t^2 - \varphi^2)}, \quad \frac{t}{t(t^2 - \varphi^2)}, \quad \frac{t^2}{t(t^2 - \varphi^2)}$$

separately. The reason for this is that every element of the matrix $(tE - A)^{-1}$ is a linear function of these three. Now

$$\frac{1}{t(t^2-\varphi^2)} = \frac{1}{\varphi^2}\left[-\frac{1}{t}+\frac{1}{2}\left(\frac{1}{t-\varphi}+\frac{1}{t+\varphi}\right)\right],$$

$$\frac{t}{t(t^2-\varphi^2)} = \frac{1}{2\varphi}\left(\frac{1}{t-\varphi}-\frac{1}{t+\varphi}\right),$$

$$\frac{t^2}{t(t^2-\varphi^2)} = \frac{1}{2}\left(\frac{1}{t-\varphi}+\frac{1}{t+\varphi}\right).$$

The first is therefore to be replaced by

$$-\frac{1}{\varphi^2}+\frac{1}{2\varphi^2}\left(\frac{\varphi}{1-e^{-\phi}}+\frac{\varphi}{e^{\phi}-1}\right) = \frac{1}{2\varphi^2}\,\frac{\varphi(e^{\phi}+1)-2}{e^{\phi}-1} = \tfrac{1}{2}\chi, \text{ say},$$

the second is to be replaced by

$$\frac{1}{2\varphi}\left(\frac{\varphi}{1-e^{-\phi}}-\frac{\varphi}{e^{\phi}-1}\right) = \tfrac{1}{2}$$

and the third, by

$$\tfrac{1}{2}\left(\frac{\varphi}{1-e^{-\phi}}+\frac{\varphi}{e^{\phi}-1}\right) = \tfrac{1}{2}\varphi\,\frac{e^{\phi}+1}{e^{\phi}-1} = \tfrac{1}{2}\omega, \text{ say}.$$

I note that $\chi = \psi/\varphi^2$ in Slocum's notation. Thus

$$f(A) = \begin{bmatrix} \tfrac{1}{2}(\omega-a_2)+a_1a_3\chi & \tfrac{1}{2}(a_1-a_1a_2\chi) & a_1^2\chi \\ -a_3+a_2a_3\chi & \tfrac{1}{2}(\omega-a_2^2\chi) & a_1+a_1a_2\chi \\ a_3^2\chi & -\tfrac{1}{2}(a_3+a_2a_3\chi) & \tfrac{1}{2}(\omega+a_2)+a_1a_3\chi \end{bmatrix}.$$

Hence, by Slocum's theorem, the corresponding infinitesimal generators of the parameter group are found by reading *down* the columns of $f(A)$. That is, the generators are denoted by

$$A_1 = [\tfrac{1}{2}(\omega-a_2)+a_1a_3\chi]\frac{\partial}{\partial a_1}+(-a_3+a_2a_3\chi)\frac{\partial}{\partial a_2}+a_3^2\chi\frac{\partial}{\partial a_3},$$

$$A_2 = \tfrac{1}{2}(a_1-a_1a_2\chi)\frac{\partial}{\partial a_1}+\tfrac{1}{2}(\omega-a_2^2\chi)\frac{\partial}{\partial a_2}+\tfrac{1}{2}(-a_3-a_2a_3\chi)\frac{\partial}{\partial a_3},$$

$$A_3 = a_1^2\chi\frac{\partial}{\partial a_1}+(a_1+a_1a_2\chi)\frac{\partial}{\partial a_2}+[\tfrac{1}{2}(\omega+a_2)+a_1a_3\chi]\frac{\partial}{\partial a_3},$$

which is Slocum's result, except that the coefficient of $\frac{\partial}{\partial a_2}$ in A_3 has a superficial difference, which disappears on using the equation $\varphi^2 = a_2^2 - 4a_1a_3$.

In order to exhibit my method in a more complicated case, I selected two of Slocum's results (type IV of the three-parameter groups; and type III of class B of the four-parameter groups). Here, however, our results did not agree; and I venture to give some details of my calculations, for the purpose of ready comparison.

Take first the three-parameter group with

$$(X_1, X_2) = 0, \quad (X_1, X_3) = X_1, \quad (X_2, X_3) = X_1 + X_2;$$

then I find the matrix

$$A = \begin{bmatrix} -a_3 & -a_3 & a_1 + a_2 \\ 0 & -a_3 & a_2 \\ 0 & 0 & 0 \end{bmatrix},$$

and so the matrix $(tE - A)^{-1}$ takes the form

$$\begin{bmatrix} \frac{1}{t + a_3} & \frac{-a_3}{(t + a_3)^2} & \frac{a_2}{(t + a_3)^2} + \frac{a_1}{a_3}\left(\frac{1}{t} - \frac{1}{t + a_3}\right) \\ 0 & \frac{1}{t + a_3} & \frac{a_2}{a_3}\left(\frac{1}{t} - \frac{1}{t + a_3}\right) \\ 0 & 0 & \frac{1}{t} \end{bmatrix}.$$

According to rule we are to put

$$1 \quad \text{in place of} \quad 1/t,$$

$$n = a_3/(e^{a_3} - 1) \quad \text{in place of} \quad 1/(t + a_3),$$

$$(n/a_3)(n + a_3 - 1) \quad \text{in place of} \quad 1/(t + a_3)^2,$$

and then the last matrix becomes $f(A)$.

It is perhaps unnecessary to write out $f(A)$ in full; but we deduce the generators

$$A_1 = n\frac{\partial}{\partial a_1}, \quad A_2 = -n(n + a_3 - 1)\frac{\partial}{\partial a_1} + n\frac{\partial}{\partial a_2}$$

$$A_3 = \left[\frac{a_1}{a_3}(1-n) + \frac{na_2}{a_3}(n+a_3-1)\right]\frac{\partial}{\partial a_1} + \frac{a_2}{a_3}(1-n)\frac{\partial}{\partial a_2} + \frac{\partial}{\partial a_3},$$

of which the third does not agree with Slocum's (the point of difference being the coefficient of $\frac{\partial}{\partial a_1}$). It may be remarked that these operators satisfy the relations,*

$$(A_1, A_2) = 0, \quad (A_1, A_3) = A_1, \quad (A_2, A_3) = A_1 + A_2,$$

and that the third of these is *not* satisfied by Slocum's operators. Indeed, it seems impossible for his operators to form a group; as (A_2, A_3) is not expressible in terms of A_1, A_2, A_3, when his symbols are used.

Next take the four-parameter group (type III on page 165, l. c.).

$$(X_1, X_2) = 0, \quad (X_1, X_3) = 0, \quad (X_2, X_3) = 0,$$
$$(X_1, X_4) = X_1, \quad (X_2, X_4) = X_1 + X_2, \quad (X_3, X_4) = X_2 + X_3.$$

Here I find the matrix $(tE - A)^{-1}$ in the form

$$\begin{bmatrix} \frac{1}{(t+a_4)} & \frac{-a_4}{(t+a_4)^2} & \frac{a_4^2}{(t+a_4)^3} & -\frac{a_3a_4}{(t+a_4)^3} + \frac{a_2}{(t+a_4)^2} + \frac{a_1}{a_4}\left(\frac{1}{t} - \frac{1}{t+a_4}\right) \\ 0 & \frac{1}{t+a_4} & \frac{-a_4}{(t+a_4)^2} & \frac{a_3}{(t+a_4)^2} + \frac{a_2}{a_4}\left(\frac{1}{t} - \frac{1}{t+a_4}\right) \\ 0 & 0 & \frac{1}{t+a_4} & \frac{a_3}{a_4}\left(\frac{1}{t} - \frac{1}{t+a_4}\right) \\ 0 & 0 & 0 & \frac{1}{t} \end{bmatrix}.$$

Proceeding according to the rule, we now have the generators

$$A_1 = n\frac{\partial}{\partial a_1}, \quad A_2 = -n(n+a_4-1)\frac{\partial}{\partial a_1} + n\frac{\partial}{\partial a_2},$$

* Which are, as they ought to be, the same as the structural relations of the given group; this is a known property of parameter groups (Lie-Engel, Transformationsgruppen, vol. 1, p. 407).

$$A_3 = [n^2(n-1) + \tfrac{3}{2}n^2a_4 - na_4 + \tfrac{1}{2}na_4^2]\frac{\partial}{\partial a_1} - n(n + a_4 - 1)\frac{\partial}{\partial a_2} + n\frac{\partial}{\partial a_3},$$

$$A_4 = \Big[-n^2(n-1)\frac{a_3}{a_4} - \tfrac{3}{2}n^2a_3 + na_3 - \tfrac{1}{2}na_3a_4 + \frac{a_1}{a_4}(1-n) + \frac{a_2}{a_4}n(n + a_4 - 1)\Big]\frac{\partial}{\partial a_1} + \Big[\frac{a_3}{a_4}n(n + a_4 - 1) + \frac{a_2}{a_4}(1-n)\Big]\frac{\partial}{\partial a_2} + \frac{a_3}{a_4}(1-n)\frac{\partial}{\partial a_3} + \frac{\partial}{\partial a_4}.$$

where $$n = a_4/(e^{a_4} - 1).$$

Here, comparing with Slocum's, the coefficients of $\frac{\partial}{\partial a_1}$ in A_2 and A_3 are of the opposite sign in the two calculations; while the coefficients of $\frac{\partial}{\partial a_1}$, $\frac{\partial}{\partial a_2}$ are almost entirely different in the two expressions for A_4.

I find that, with my results,

$$(A_1, A_2) = 0, \quad (A_1, A_3) = 0, \quad (A_2, A_3) = 0,$$

$$(A_1, A_4) = A_1, \quad (A_2, A_4) = A_1 + A_2, \quad (A_3, A_4) = A_2 + A_3,$$

which, of course, ought to be the case, on account of the general properties of the parameter group (*i. e.*, it has the same structural constants as the given group).

Taking Slocum's operators, I did not succeed in expressing (A_2, A_4) in terms of A_1, A_2 A_3, A_4, ; and it may be doubted if these operators can be generators of a group. But I did not calculate (A_3, A_4), for the evaluation of this alternant proved very laborious, when I used the operators found above, and it seemed superfluous to apply any further tests of this character.

§ 4. Having found these differences between my work and Slocum's, it seemed worth while to examine all the types; and I found the following divergences (in addition to those given before).

Class A (of the Four-Parameter Groups).

Type II.—The coefficient of $\frac{\partial}{\partial a_1}$ in A_3 comes out as

$$\frac{a_2[1 - e^{a_4\beta} + \beta(1 - e^{-a_4})e^{a_4\beta}]}{(e^{a_4\beta} - 1)(e^{a_4(\beta-1)} - 1)},$$

and that in A_4 is

$$\frac{a_1}{a_4}\left(1 - \frac{a_4\beta}{e^{a_4\beta} - 1}\right) + \frac{a_2 a_3}{a_4(\beta-1)}\left[\frac{1}{e^{a_4}-1} + \frac{\beta(\beta-2)}{e^{a_4\beta} - 1} - \frac{(\beta - 1)^2}{e^{a_4(\beta-1)}-1}\right].$$

Type III.—The first term in A_3 appears as*

$$\left\{\frac{2a_3}{(e^{a_4} - 1)^2}[(a_4 - 1)e^{a_4} + 1] + \frac{a_2 - 2a_3}{e^{a_4} + 1}\right\}\frac{\partial}{\partial a_1},$$

while the first and second in A_4 are

$$\frac{1}{a_4}\left\{a_1\left(1 + \frac{a_4}{e^{a_4} - 1}\right) + 2a_3^2\left[\frac{(a_4-1)e^{a_4}+1}{(e^{a_4} - 1)^2} - \frac{1}{e^{a_4} + 1}\right]\right\}\frac{\partial}{\partial a_1}$$
$$+ \left[\frac{a_2}{a_4}\left(1 - \frac{a_4}{e^{a_4} - 1}\right) + 2a_3\frac{(a_4 - 1)e^{a_4} + 1}{(e^{a_4} - 1)^2}\right]\frac{\partial}{\partial a_2}.$$

Type IV.—The coefficient of $\frac{\partial}{\partial a_1}$ in A_2 comes out as

$$-\frac{a_3[e^{a_4}(a_4 - 1) + 1]}{(e^{a_4} - 1)^2} = -\frac{na_3}{a_4}(n + a_4 - 1),$$

and that in A_4 as

$$\frac{a_1}{a_4}(1 - n) + \frac{a_2 a_3}{a_4^2}(n^2 + a_4 n - 1),$$

where

$$n = a_4/(e^{a_4} - 1).$$

Class B (of the Four-Parameter Groups).

Type II.—The coefficient of $\frac{\partial}{\partial a_2}$ in A_3 appears to be equal, but of the opposite sign, to Slocum's; while that in A_4 is

$$\frac{a_2}{a_4}(1 - n) + \frac{na_3}{a_4\beta}(n + a_4\beta - 1),$$

where

$$n = a_4\beta/(e^{a_4\beta} - 1).$$

Type III.—See § 3, above.

Type V.—The coefficient of $\frac{\partial}{\partial a_2}$ in A_4 comes out as

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* Although I agree with Slocum's expression for A_2 in this type, yet it seems neater to put the coefficient of $\frac{\partial}{\partial a_1}$ in the shape $\left(-\frac{a_3}{e^{a_4}+1}\right)$.

$$\frac{a_2}{a_4}\left(1-\frac{a_4}{e^{a_4}-1}\right)+\frac{a_3[(a_4-1)\,e^{a_4}+1]}{(e^{a_4}-1)^2}.$$

Owing to the large amount of arithmetic involved in the deduction of these results, I can hardly hope to have carried them through without error. Unfortunately, I have not been able to obtain any assistance in verifying the work; and all my calculations have had to be made under the pressure of professorial class-work. However, in all cases where differences showed themselves between my first results and those of Slocum, my work has been carried out twice (and, in some cases, three times); and all checks have been applied that suggested themselves (see, for instance, § 5 below).

§ 5. It may be of use to remark that if the matrix A belonging to one group can be obtained from that of a second group by specializing a certain constant, then it is possible to obtain the matrix $f(A)$ of the first group, from that of the second, by using the same specialized constant, *provided that this change does not affect the invariant factors of* $|tE-A|$. As the parameter operators depend only on $f(A)$, it follows that the parameter operators of the first given group must be deducible similarly from that of the second group.

Thus, taking the three-parameter groups, type III has a matrix A which is deduced from that of type II by writing $\beta=1$; and this does not modify the invariant factors of $|tE-A|$; hence the parameter operators of III follow from those of II by putting $\beta=1$. Similarly V comes from II, by putting $\beta=0$.

So also, in Class A of the four-parameter groups, type IV comes from II, by writing $\beta=1$; that is, if the alterations mentioned in § 4 are made first; Slocum's results are inconsistent here. Further, in class B, types IV and VI come from I, by putting (i) $\beta=\alpha$, and (ii) $\gamma=\beta=\alpha=1$. Also, type V is obtained from II by writing $\beta=\alpha=1$; but here Slocum's operators must be modified (see § 4) before this can be verified.

§ 6. Concluding, it may be added that it is also possible to find $f(A)$ directly from the series in which we can expand the function

$$A/[E-\exp(-A)]=E+\tfrac{1}{2}A+\tfrac{1}{12}A^2+\cdots.$$

To do so, we must make use of the characteristic equation * of the matrix A, in order to express the higher powers of A. This leads to tedious work unless the characteristic equation reduces to one in two terms only; for instance, types V and VI (in Slocum's list of three-parameter groups) lead to the equations for A

$$A^2 + a_3 A = 0, \quad A^2 = 0$$

respectively. This method has been used by Dr. H. F. Baker in a recent paper on the calculation of the finite equations of a group from its structural constants (*Proceedings of the London Mathematical Society*, volume 34 (1902), page 91); but Baker's work relates solely to the determination of the matrix which is denoted by B in my notation, *i. e.*, the matrix reciprocal to $f(A)$.

I have not actually applied the last method to any of the harder cases; indeed, I have only used it for the two cases just mentioned, when it gives

Type V. $$f(A) = E + \left(\frac{1}{a_3} - \frac{1}{e^{a_3} - 1}\right) A.$$

Type VI. $$f(A) = E + \tfrac{1}{2}A.$$

It may, however, prove useful as an alternative means of verification.

QUEEN'S COLLEGE, GALWAY,
February 25, 1902.

ON THE PARABOLAS (OR PARABOLOIDS) THROUGH THE POINTS COMMON TO TWO GIVEN CONICS (OR QUADRICS).

BY PROFESSOR T. J. I'A. BROMWICH.

(Read before the American Mathematical Society, April 26, 1902.)

IN the December issue of the BULLETIN (page 122, December, 1901) Huntington and Whittemore have called attention to the features of conics which touch the line infinity

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* This is obtained as follows: Let $\phi(t)$ be the quotient of the determinant $|tE - A|$ by the highest common factor of all its first minors. Then the equation is

$$\phi(A) = 0.$$

See Frobenius, *Crelle*, vol. 84; and other references given by the present author in a review, BULLETIN, vol. 7 (1900), p. 308.