ON THE ABELIAN GROUPS WHICH ARE CONFORMAL WITH NON-ABELIAN GROUPS.

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Two distinct groups are said to be conformal when they contain the same number of operators of each order.* The present paper is devoted to the determination of all the abelian groups which are conformal with non-abelian groups. The complete solution of the converse of this problem, *viz.*, the determination of all the non-abelian groups which are conformal with abelian ones is much more difficult, since a large number of distinct non-abelian groups may be conformal with the same abelian group while no more than one abelian group can be conformal with one non-abelian group. In fact, two distinct abelian groups cannot be conformal.

It is well known that there is only one group of order 2^m which does not include any operator of order 4, *viz.*, the group of type $(1, 1, 1, \cdots\cdots)$.† Moreover, there is only one cyclic group of order 2^m, and when $m < 4$ no two groups of order 2^m are conformal. We proceed to prove that every abelian group G of order 2^m which does not satisfy one of these conditions is conformal with at least one non-abelian group.

Let H be the subgroup of G which is generated by the square of one of its independent generators s of lowest order together with all the other independent generators of G. The order of H is 2^{m-1}. Since $m > 3$ there is an operator t of order 2 which has the following properties.‡ It transforms H into itself, it is commutative with half of the operators of H (including all those which are not of highest order), and it transforms the rest into themselves multiplied by an operator of order 2 which is not the square of a non-invariant operator of H; *i. e.*, t does not transform an operator of order 4 contained in H into its inverse. The non-abelian group generated by H and t is conformal with G whenever $s^2 = 1$.

When the order of s exceeds two, we may make the group generated by t and H (written as a regular substitution group) simply isomorphic with itself by writing it in two

* *Quar. Jour. of Math.*, vol. 28 (1896), p. 270.

† *Ibid.*, p. 208.

‡ BULLETIN, vol. 5 (1898), p. 245; also vol. 6 (1899), p. 236.

distinct sets of letters.* If in this intransitive group t is replaced by the continued product of t, the substitution of order two which merely permutes corresponding letters of the two systems of intransitivity, and s^2 in one of the systems of letters there results a transitive group which is conformal with G. That is, any abelian group of order 2^m, $m > 3$, which is neither cyclic nor of type $(1, 1, 1, \cdots (m \text{ times}))$ is conformal with at least one non-abelian group.

It will now be assumed that the order of G is p^m (p being an odd prime number and $m > 2$) and that G is non-cyclic. Let H be the subgroup generated by s^p (s being one of the independent generators of lowest order in G) together with all the other independent generators of G. There is an operator t of order p which transforms H into itself, is commutative with each of its operators contained in a subgroup of order p^{m-2}, and transforms the rest into themselves multiplied by invariant operators of order p. This t and H generate a group conformal with G whenever $s^p = 1$; for if s_1 is any substitution of H that is not commutative with t it is easy to see that $(ts_1)^p = ts_1ts_1 \ldots (p \text{ times}) = ts_1t^{-1}t^2s_1t^{-2}t^3s_1t^{-3}t^4 \cdots t^{1-p}t^ps_1 = s_1^p$. †

When s^p differs from identity the group generated by H and t, written as a regular group, may be made simply isomorphic with itself $p - 1$ times by writing each substitution in p distinct sets of letters, and t may be replaced by the continued product of t, the substitution of order p which merely permutes the corresponding letters of these systems of intransitivity, and the pth power of s in one of these systems. In the resulting group the pth power of the operators will be the same as those of G taken in the same order and hence t will be conformal with G.‡

If a non-abelian group whose order is not some power of a prime is conformal with an abelian group G, it must be the direct product of its subgroups whose orders are powers of single primes, and hence each of these subgroups is conformal with an abelian group and at least one of these is non abelian. From what precedes it may be observed that *the necessary and sufficient conditions that any abelian group of order* $2^{\alpha_0}p_1^{\alpha_1}p_2^{\alpha_2}\cdots$ ($p_1, p_2, \cdots$ *being distinct odd primes*) *is conformal with at least one non-abelian group are:* 1° *at least one of its subgroups of orders* $2^{\alpha_0}, p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots$ *is non-cyclic;* 2° *if the order* $p_\beta^{\alpha_\beta}$ *of this subgroup is odd then* $\alpha_\beta > 2$, *if the order*

* *Quar. Jour. of Math.*, vol. 28 (1896), p. 236.

† *Transactions of the Am. Math. Society*, vol. 2 (1901), p. 262.

‡ *Transactions of the Am. Math. Society*, vol. 2 (1901), p. 264.

is even (2^{a_0}) *then the subgroup must involve operators of order* 4 *and* $a_0 > 3$. Since any number of these factors may be non-abelian, there cannot be an upper limit to the number of non-abelian groups which may be conformal with one abelian group. This fact may be seen in many other ways.

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THE INFINITESIMAL GENERATORS OF CERTAIN PARAMETER GROUPS.

BY DR. S. E. SLOCUM.

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BY means of the r independent infinitesimal transformations

$$X_j \equiv \sum_1^n{}_k \, \xi_{jk}(x_1, \cdots, x_n) \frac{\partial}{\partial x_k} \qquad (j = 1, 2, \cdots, r)$$

we may construct a family of transformations

$$(1) \qquad x_i' = f_i(x_1, \cdots, x_n, a_1, \cdots, a_r) \qquad (i = 1, 2, \cdots, n)$$

with r essential parameters $a_1, \cdots, a_r$, where $f_i(x, a)$ is defined in the neighborhood of the identical transformation by the series

$$f_i(x, a) \equiv x_i + \sum_1^r{}_j \, a_j X_j x_i + \frac{1}{2!} \sum_1^r{}_j \sum_1^r{}_k X_j X_k x_i + \cdots$$
$$(i = 1, 2, \cdots, n).$$

The transformations defined by these equations for assigned values of the a's may be denoted by T_a. Let the differential operators X_j ($j = 1, 2, \cdots, r$) satisfy Lie's criterion, that is, let

$$X_j X_k - X_k X_j \equiv \sum_1^r{}_s \, c_{jks} X_s \qquad (j, k = 1, 2, \cdots, r).$$

Then by Lie's chief theorem, the family of transformations T_a, defined by equations (1), forms a group G.* Conse-

* Continuierliche Gruppen, pp. 390–391.