BULLETIN OF THE

AMERICAN MATHEMATICAL SOCIETY

THE EIGHTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE Eighth Summer Meeting, supplemented by the Third Colloquium, of the AMERICAN MATHEMATICAL SOCIETY was held at Cornell University, Ithaca, N. Y., through the week August 19–24, 1901. The four sessions of the Summer Meeting proper occupied the first two days of the week. Only an unduly severe compression made it possible to complete the lengthy programme in this period. More time, especially for the discussion of papers, is needed at the larger meetings of the year, and this question is to receive immediate attention. On the present occasion various circumstances made an adequate provision of time practically impossible.

The attendance at the Summer Meeting numbered about sixty persons, including the following forty-five members of the Society:

Dr. G. A. Bliss, Professor Oskar Bolza, Professor E. W. Brown, Dr. W. G. Bullard, Professor F. N. Cole, Professor L. L. Conant, Mr. A. T. DeLury, Professor L. W. Dowling, Professor L. E. Dickson, Professor W. P. Durfee, Professor H. T. Eddy, Professor John Eiesland, Professor Peter Field, Dr. J. C. Fields, Professor T. S. Fiske, Mr. W. B. Ford, Mr. B. F. Groat, Dr. E. R. Hedrick, Professor T. F. Holgate, Dr. J. I. Hutchinson, Dr. Edward Kasner, Mr. C. J. Keyser, Dr. G. H. Ling, Professor T. E. McKinney, Professor James McMahon, Professor W. H. Maltbie, Professor J. L. Markley, Professor W. H. Metzler, Dr. G. A. Miller, Professor E. H. Moore, Dr. G. A. Murray, Professor G. D. Olds, Professor W. F. Osgood, Mr. F. G. Radelfinger, Professor E. D. Roe, Professor E. B. Skinner, Dr. Virgil Snyder, Dr. H. F. Stecker, Professor J. H. Tanner, Professor E. J. Townsend, Professor E. B. Van Vleck, Professor L. A. Wait, Professor L. G. Weld, Dr. J. v. E. Westfall, Professor Alexander Ziwet.

The President of the Society, Professor Eliakim Hastings Moore, occupied the chair, being relieved by Vice-President Professor T. S. Fiske. An address of welcome by Profes sor L. A. Wait in behalf of the University, was the forerunner of a most generous hospitality accorded the Society by the University and its individual officers. Resolutions adopted at the closing session partially express the Society's sense of appreciation of the courtesies extended to it throughout its stay in Ithaca.

The Council announced the election of the following persons to membership in the Society: Dr. E. R. Hedrick, Yale University; Professor S. W. Reaves, Clemson College, S. C., Twelve applications for admission to the Society were received. A committee consisting of the President, Professor Fiske, and Professor Osgood, was appointed to prepare and submit to the Council at the October meeting a list of nominations of officers of the Society for the coming year.

The following papers were presented at the meeting:

(1) Professor MAXIME BÔCHER: "On certain pairs of transcendental functions whose roots separate each other."

(2) Dr. J. I. HUTCHINSON: "On a class of automorphic functions."

(3) Professor A. PRINGSHEIM: "Ueber den Goursat'schen Beweis des Cauchy'schen Integralsatzes."

(4) Professor A. PRINGSHEIM: "Ueber die Anwendung der Cauchy'schen Multiplicationsregel auf bedingt convergente oder divergente Reihen."

(5) Mr. W. B. FORD: "On the expression of Bessel functions in terms of the trigonometric functions."

(6) Professor E. H. MOORE: "On the theory of improper definite integrals."

(7) Professor OSKAR BOLZA: "New proof of a theorem of Osgood in the calculus of variations."

(8) Dr. G. A. BLISS: "The problem of the calculus of variations when the end point is variable."

(9) Dr. J. C. FIELDS: "On certain relations existing between the branch points and the double points of an algebraic curve."

(10) Dr. J. C. FIELDS: "The Riemann-Roch theorem and the independence of the conditions of adjointness in the case of a curve for which the tangents at the multiple points are distinct from one another."

(11) Professor E. B. VAN VLECK: "A proof of the convergence of the Gaussian continued fraction for

$$\frac{F(\alpha, \beta + 1, \gamma + 1, x)}{F(\alpha, \beta, \gamma, x)}."$$

(12) Professor T. E. McKINNEY: "Some new kinds of continued fractions."

(13) Professor E. D. ROE: "Note on symmetric functions."

(14) Dr. G. A. MILLER: "Groups defined by the orders of two generators and the order of their product."

(15) Dr. H. F. STECKER: "On the determination of surfaces capable of conformal representation upon the plane so that the geodetic lines shall be represented by a prescribed system of plane curves."

(16) Dr. C. N. HASKINS: "On the invariants of quadratic differential forms."

(17) Dr. EDWARD KASNER: "The cogredient and digredient theories of double binary forms."

(18) Professor MAXIME BÔCHER: "On Wronskians of functions of a real variable."

(19) Mr. F. G. RADELFINGER: "The analytical representation of a multiform function in the domain of an isolated singular point."

(20) Dr. VIRGIL SNYDER: "On the forms of unicursal sextic scrolls with a multiple linear directrix and one double curve."

(21) Dr. H. F. STECKER: "Concerning the osculating plane of m-fold space filling curves of the Hilbert-Moore type."

(22) Dr. H. F. STECKER: "On non-euclidean properties of plane cubics and of their first and second polars."

(23) Professor L. E. DICKSON: "Theory of linear groups in an arbitrary field."

(24) Mr. H. L. RIETZ: "On primitive groups of odd composite order."

(25) Miss I. M. SCHOTTENFELS: "On the non-isomorphism of two simple groups of order $8!/2$."

(26) Professor L. W. DOWLING: "On the generation of plane curves, of any order higher than four, with four double points."

(27) Professor L. E. DICKSON: "The configuration of the 27 lines on a cubic surface and the 28 bitangents to a quartic curve."

(28) Professor E. H. MOORE: "Concerning the second mean value theorem of the integral calculus."

(29) Mr. I. E. RABINOVITCH: "The application of circulants to the solution of algebraic equations."

(30) M. EMILE LEMOINE: "Note sur la construction approchée de π de Mr. George Peirce."

(31) Dr. C. W. McG. BLACK: "The parametric repre-

sentation of the neighborhood of a singular point of an analytic surface."

(32) Professor ALEXANDER PELL: "Some remarks on surfaces whose first and second fundamental forms are the second and first respectively of another surface."

Professor Pringsheim's papers were presented to the Society through Professor Moore, Dr. Haskins's through Professor Osgood, Mr. Rietz's through Dr. Miller, Dr. Rabinovitch's through Professor Morley, M. Lemoine's through the Secretary. In the absence of the authors the first paper of Professor Bôcher and the papers of Dr. Haskins and Dr. Black were read by Professor Osgood, Professor Pringsheim's papers by Professor Moore, M. Lemoine's paper by Professor Brown. The second paper of Professor Bôcher, the papers of Mr. Rietz, Miss Schottenfels, and Professor Pell, and the second paper of Professor Moore were read by title.

In Professor Bôcher's first paper the following theorem is proved by making use of the Riccati's equations satisfied by the ratios of the functions $\varphi_2 y' - \varphi_1 y$ and $\psi_2 y' - \psi_1 y$: If, in a certain interval, $p, q, \varphi_2, \varphi_1, \psi_2, \psi_1$ are continuous real functions of the real variable x, and if the last four of these functions have continuous derivatives, then, y being a solution not identically zero of the differential equation

$$y'' + py' + qy = 0,$$

the roots of the functions

$$\varphi_2 y' - \varphi_1 y, \quad \psi_2 y' - \psi_1 y$$

will separate each other if no one of the three functions

$$\varphi_1\psi_2 - \varphi_2\psi_1, \quad \varphi_1'\varphi_2 - \varphi_1\varphi_2' + \varphi_1^2 + p\varphi_2\varphi_1 + q\varphi_2^2,$$
$$\psi_1'\psi_2 - \psi_1\psi_2' + \psi_1^2 + p\psi_2\psi_1 + q\psi_2^2$$

vanishes at any point of the interval in question.

Certain extensions of this theorem are established and some important special cases are considered. The relations of the subject to Sturm's paper in *Liouville's Journal*, volume 1 (particularly pp. 149–164), and to earlier papers of the author are discussed.

The paper by Dr. Hutchinson considers the functions which are automorphic for the monodromic group of the Riemann surface $w^3 = (z - \varkappa_1)(z - \varkappa_2)(z - \mu_1)^2(z - \mu_2)^2$. The moduli of periodicity of the integrals of the first kind are expressible in terms of a single parameter ζ the group of transformations on which is generated by the two

$$S: \quad \zeta' = \zeta + \sqrt{3}, \qquad T: \quad \zeta' = -1/\zeta.$$

The automorphic functions of ζ have for natural boundary the axis of reals, and for a fundamental region one limited by the unit circle and two parallels to the imaginary axis at the distances $\pm \frac{1}{2}\sqrt{3}$. Every such function is rationally expressible in terms of any one of the Rosenhain moduli.

In his first paper Professor Pringsheim recurs to the important lemma of Goursat (*Transactions*, volume 1 (1900), No. 1) and gives in detail its application to the proof of Goursat's form of Cauchy's theorem. Of this application Goursat gave a brief indication, which was subject to misinterpretation. The present paper contains also critical remarks concerning the possible crinkliness of the curve of integration of the integral if the theorem holds by this proof, as well as a more general form of the lemma valid and available in some cases of greater crinkliness.

In his memoir "Über die Multiplication bedingt convergenter Reihen" (*Mathematische Annalen*, volume 21 (1883), p. 327) Professor Pringsheim pointed out that the product series resulting from the application of Cauchy's rule of multiplication to two conditionally convergent series or to a conditionally convergent and a divergent series may be be unconditionally convergent, and suggested the query whether the same conclusion might hold for the product series of two conditionally convergent series. Professor Cajori (*Transactions*, volume 2 (1901), No. 1) showed that this may happen for the product series of two conditionally convergent series and also for the product series of a conditionally convergent and a divergent series. In his second paper Professor Pringsheim proves that all these results and the similar result for the product series of two divergent series are precisely typical for large categories of cases which are obtained by extremely simple methods depending primarily upon the properties of a power series on its circle of convergence.

In studying the convergence of any one of the well-known developments for an arbitrary function $f(x)$ in terms of Bessel functions (developments such as occur for instance in the study of mathematical physics) it becomes desirable to have some simple relation connecting $J_n(z)$ with the trigonometric functions, $J_n(z)$ being here used to represent Bessel's function of order n. Mr. Ford's paper in dealing with the relationship between the two classes of functions establishes two theorems which are of especial value in connection with the study of series developments such as have just been mentioned, though they may be used to advantage in any study relative to the behavior of $J_n(z)$ in the neighborhood of the point $z = \infty$.

In his first paper Professor Moore considers the various current types of improper simple definite integrals (connected with the names of (1) Cauchy, Riemann, du Bois-Reymond, Dini, Schoenflies; (2) Harnack, Jordan; (3) Hölder; (4) de la Vallée-Poussin) and, generalizing their diversities, defines a system of types, which differ according to the way in which the integral depends definitionally in its limiting process or processes upon the set of points of singularity of the integrand function with respect to definite integration. The properties of the general type of this system are investigated.

Professor Bolza's paper gives a simple proof of Professor Osgood's theorem on a characteristic property of a strong minimum in the calculus of variations (*Transactions*, July, 1901). The paper will be published in the *Transactions*.

The paper by Dr. Bliss concerns the problem of the calculus of variations in which one end point is restricted to lie upon a given curve while the other is fixed. Sufficient conditions for a maximum or minimum in this case are derived by Kneser in his Variationesrechnung and some of them are also proved to be necessary. The purpose of this paper was to find a complete set of necessary and sufficient conditions by the method which Weierstrass used for fixed end points. By this method the necessity of the conditions was proven for Kneser's exceptional case when the fixed end point lies beyond the "Brennpunkt," at which, besides, the envelope of the field has a multiple point of a particular kind. It cannot, however, be applied when "Brennpunkt" and end point are coincident. From the discussion it follows that the "Brennpunkt" always lies nearer to the given curve than the conjugate point.

Dr. Field's first paper is in abstract as follows: Let $F(z, u) = 0$ be the equation of an irreducible algebraic curve of degree n. The multiple points of the curve we shall assume to be double points with distinct tangents. Further we shall assume that no tangent at a double point, no inflexional tangent, and no asymptote is parallel to either of the axes, and also that the asymptotes are all distinct from one another, and no two parallel to each other. The coefficient of u'' we shall take equal to 1. In § 2 of his paper "On the reduction of the general abelian integral,"* the author has given the formula

$$\frac{du}{dz}=\sum_{\lambda}\frac{\gamma_\lambda F(a_\lambda, u)}{(z-a_\lambda)(u-b_\lambda)},$$

where the summation extends to all branch and double points (a_λ, b_λ), u being regarded as dependent variable, and where

$$\gamma_\lambda=\frac{1}{[F_u'']_{a_\lambda, b_\lambda}} \quad \text{or} \quad \frac{2}{[F_u'']_{a_\lambda, b_\lambda}}$$

according as (a_λ, b_λ) is a branch point or a double point.

On reducing the above formula to the form

$$\begin{aligned}\frac{du}{dz}&=\sum_{\lambda}\frac{\gamma_\lambda F(a_\lambda, u)}{(z-a_\lambda)(u-b_\lambda)}\\&=z^{-1}p_1(z, u)+z^{-2}p_2(z, u)+\cdots+z^{-k}p_k(z, u)+\cdots,\end{aligned}$$

where $p_1(z, u), p_2(z, u), \cdots$ are homogeneous polynomials in z, u of degree $n-1$, we readily see that the coefficients of all the terms in the polynomials $p_1(z, u), \cdots, p_{n-2}(z, u)$ must vanish and that $z^{-(n-1)}p_{n-1}(z, u)$ must reduce to the single term $z^{-1}u$, so that we have

$$\frac{du}{dz}=z^{-1}u+z^{-n}p_n(z, u)+\cdots$$

The conditions implied upon the constants γ_λ by the vanishing of the coefficients here in question may be shown to be

$$\sum_{\lambda}\gamma_\lambda a_\lambda^{\,i} b_\lambda^{\,k}=0 \qquad (i+k=0, 1, 2, \cdots, n-2),$$

with the exception of the one condition corresponding to the pair of values $i=0$, $k=n-2$, which takes the form

$$\sum_{\lambda}\gamma_\lambda b_\lambda^{\,n-2}=1.$$

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* *Trans. Amer. Math. Soc.*, vol. 2 (1901), pp. 49–86.

These conditions may be included in the single formula

$$\sum_{\lambda} \gamma_{\lambda} G(a_{\lambda}, b_{\lambda}) = d_{0, n-2}$$

where

$$G(z, u) = \sum d_{i, k} z^i u^k \qquad (i + k \leqq n - 2)$$

is an arbitrary polynomial in z, u of degree $n - 2$.

On substituting their values for the quantities γ_{λ} we obtain a formula which may conveniently be written in the form

$$\sum_{\lambda} \frac{G(a_{\lambda}, b_{\lambda})}{[F_u'']_{a_{\lambda}, b_{\lambda}}} = d_{0, n-2}$$

on supposing an element of the summation corresponding to a double point to be repeated.

A similar formula is obtained on regarding z as the dependent variable.

Dr. Field's second paper is summarized as follows: Let $F(z, u) = 0$ be the equation to an irreducible algebraic curve of degree n. It is supposed that the tangents at the multiple points are distinct from one another and that these points are not at the same time branch points, and further it is assumed that the asymptotes are all distinct from one another, none of them parallel to the axis of u, and no two parallel to each other.

In a paper presented at the meeting of the Chicago Section in December last, the author proved the Riemann-Roch theorem and the independence of the conditions of adjointness in the case where the multiple points are all double points. The present paper extends the methods there employed to multiple points of higher order.

It is shown that the most general rational function which becomes infinite only at ∞ is given by the expression

$$\sum_{\lambda} \sum_{s=0}^{\sigma_{\lambda}-2} \sum_{r=0}^{\sigma_{\lambda}-s-2} c_{r, s, \lambda} \left(\frac{\delta}{\delta a_{\lambda}}\right)^r \left(\frac{\delta}{\delta b_{\lambda}}\right)^s \frac{F(a_{\lambda}, u)}{(z - a_{\lambda})(u - b_{\lambda})} + T(z, u),$$

where $T(z, u)$ is an arbitrary polynomial and the coefficients $c_{r, s, \lambda}$ arbitrary constants, where σ_{λ} is the order of the multiple point $(a_{\lambda}, b_{\lambda})$, and where the summation extends to all the multiple points $(a_{\lambda}, b_{\lambda})$. This may be written in the form

$$\sum_{\lambda} \gamma_{\lambda} \frac{F(a_{\lambda}, u)}{(z - a_{\lambda})(u - b_{\lambda})} + T(z, u)$$

on putting

$$\gamma_\lambda = \sum_{s=0}^{\sigma_\lambda - 2} \sum_{r=0}^{\sigma_\lambda - s - 2} c_{r,s,\lambda} \left(\frac{\delta}{\delta a_\lambda}\right)^r \left(\frac{\delta}{\delta b_\lambda}\right)^s.$$

On indicating by $(a_1, b_1), \cdots, (a_\delta, b_\delta)$ the multiple points, supposed to be δ in number, and by $(a_{\delta+1}, b_{\delta+1}), \cdots, (a_{\delta+Q}, b_{\delta+Q})$ certain Q other points, we consider the conditions under which the function

$$\sum_{\lambda=1}^{\delta+Q} \gamma_\lambda \cdot \frac{F(a_\lambda, u)}{(z - a_\lambda)(u - b_\lambda)} + T(z, u)$$

remains finite at ∞, an operator γ_λ here reducing to an arbitrary constant in case (a_λ, b_λ) is an ordinary point or a double point. We find that the polynomial $T(z, u)$ must reduce to an arbitrary constant c, and on expressing our function in the form

$$\sum_{\lambda=1}^{\delta+Q} \gamma_\lambda \cdot \frac{F(a_\lambda, u) - F(a_\lambda, b_\lambda)}{(z - a_\lambda)(u - b_\lambda)} + C$$

$$= z^{-1} p_1(z, u) + z^{-2} p_2(z, u) + \cdots,$$

where the functions $p_1(z, u), p_2(z, u), \cdots$ are homogenous polynomials in (z, u) of degree $n - 1$, we see that the coefficients must all vanish in the $p - 2$ polynomials $p_1(z, u), \cdots, p_{n-2}(z, u)$. This imposes certain conditions on the constants $C_{r,s,\lambda}$ involved in the operators γ_λ, and these conditions we find to be given by the equations

$$\sum_{\lambda=1}^{\delta+Q} \gamma_\lambda \cdot a_\lambda^i b_\lambda^k = 0 \qquad (i + k = 0, 1, 2, \cdots, n - 3).$$

An operator γ_λ operates throughout on any function (a_λ, b_λ) which it may happen to precede.

From the fact that a rational function which nowhere becomes infinite must be a constant, follows in the case where $Q = 0$ that the constants involved in the operators γ_λ must all have the value 0, and the interpretation of this necessity proves the independence of the conditions of adjointness. In the case $Q > 0$ our equations show that the number of arbitrary constants $C_{r,s,\lambda}$ involved in their solutions is $Q - q$, where q is the strength of the set of Q points $(a_{\delta+1}, b_{\delta+1}), \cdots, (a_{\delta+Q}, b_{\delta+Q})$ in determining an adjoint polynomial of degree $n - 3$, and from this the Riemann-Roch theorem immediately follows.

Professor Van Vleck considered the convergence of the well-known continued fraction of Gauss for

$$G(\alpha, \beta, \gamma, x) = \frac{F(\alpha, \beta + 1, \gamma + 1, x)}{F(\alpha, \beta, \gamma, x)}$$

This has been previously shown by Thomè and Riemann to converge to $G(\alpha, \beta, \gamma, x)$ as its limit over the entire complex plane with the exception of the portion of the real axis included between $x = 1$ and $x = +\infty$. The proofs of both of these writers are, however, of a special and somewhat involved character, not based upon considerations which can be applied to prove the convergence of many other continued fractions. The object of the present paper was to demonstrate the convergence by principles of a fundamental character for the theory of algebraic continued fractions. For this purpose two new theorems are established, the first of which is as follows:

I. If in a continued fraction

$$\frac{a_1 x}{1} + \frac{a_2 x}{1} + \frac{a_3 x}{1} + \cdots$$

the greatest modulus of any point of condensation of the sequence $a_1, a_2, a_3, \ldots$ is k, then within a circle of radius $\frac{1}{4k}$, described about the origin as center, the continued fraction will represent an analytic function, and the only singularities of the function contained within the circle will be poles. In any region excluding these poles and lying in the interior of the circle the convergence will be uniform.

This theorem will be further developed and published in a separate paper. The other theorem is as follows:

II. If from and after some fixed point in the continued fraction

$$\frac{1}{b_1 z} + \frac{1}{b_2} + \frac{1}{b_3 z} + \frac{1}{b_4} + \cdots$$

b_n is real and positive, and if $\sum b_n$ is divergent, the continued fraction will converge over the entire plane of z with the exception of 1° the whole or part of the negative half of the real axis, and 2° a series of isolated points $p_1, p_2, \cdots$ which lie without this half of the axis. The limit of the continued fraction is an analytic function whose only singularities exterior to the negative half axis are the points $p_1, p_2, \ldots$ which are poles.

By applying these theorems to the continued fraction of

Gauss, its convergence can be immediately demonstrated when α, β, γ are real. In case one or more of these three elements are complex, the second theorem must first be suitably modified and may then be employed as before. The two theorems, either separately or together, can be applied to all, or nearly all, the common algebraic continued fractions.

Professor McKinney discussed certain new kinds of continued fractions. These are characterized by 1° The system of intervals into which the series of real numbers is divided; and 2° the equation connecting two consecutive complete quotients. The paper defines these continued fractions and states some laws obeyed by developments representing the square root of integers, comparing these laws with the like cases of ordinary continued fractions.

Professor Roe's paper first calls attention to other proofs, and then gives a proof of Professor W. H. Metzler, that in the product of a symmetric function

$$\sum a_1^{p_1} a_2^{p_2} \cdots a_m^{p_m}$$

by the alternant $(0, 1, 2, \cdots, m-1)$ the result is obtained by adding the p's in all possible permutations to the exponents of the columns of the alternant written as a determinant, in which each line contains the powers of a single letter, thus giving the product in the general case as the sum of $m!$ alternants. This proof consists in combining the theorem for the multiplication of a determinant of the nth order by a polynomial of n terms, with the fact that every symmetric function is a function of the s's, each of which is a polynomial of n terms.

Attention is next called to the fact that in the preceding product the coefficient of the alternant $(i_1, i_2, \cdots, i_m)$ is equal to the coefficient of $a_{i_1} a_{i_2} \cdots a_{i_m}$ in the determinant

$$\begin{vmatrix} a_{i_1} & a_{i_2} \cdots & a_{i_m} \\ a_{i_1-1} & a_{i_2-1} \cdots & a_{i_m-1} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ a_{i_1-m+1} & \cdots & a_{i_m-m+1} \end{vmatrix}$$

which determinant is represented by $\{i_1 i_2 \cdots i_m\}$, and that a symmetric function $a_0^n \sum a_1^{p_1} a_2^{p_2} \cdots a_m^{p_m}$ is equal to a sum of determinants of the preceding form of the nth order in the

a's, with a similar coefficient excepting a sign factor, so that we obtain, with certain obvious details which need not be given here, the two formulas

$$(1)\quad (0, 1, 2, \cdots, m-1)\sum a_1^{p_1} a_2^{p_2} \cdots a_m^{p_m} = \sum \left\{ \begin{matrix} p_1 p_2 \cdots p_m \\ i_1 i_2 \cdots i_m \end{matrix} \right\} (i_1\, i_2 \cdots i_m),$$

$$(2)\quad a_0^n \sum a_1^{p_1} \cdots a_m^{p_m} = \sum (-1)^{\mu} \left\{ \begin{matrix} p_1 p_2 \cdots p_m \\ i_1 i_2 \cdots i_m \end{matrix} \right\} \{ i_{m+1} i_{m+2} \cdots i_{m+n} \}.$$

Let l, m, n represent the orders of three operators L, M, N respectively and let $LM = N$. It is known that the group generated by any two of these operators is completely defined by the values of l, m, n whenever one of the following conditions is satisfied: 1° Two of these numbers are equal to 2; 2° one of them equals 2 the other 3, while the third has one of the three values, 3, 4, 5. The object of Dr. Miller's paper is to prove that two of the operators mentioned above can be so selected as to generate any one of an infinite system of groups of finite order whenever the values of l, m, n do not satisfy one of the two given conditions. This theorem is proved by observing that substitutions corresponding to L and M can be so selected as to generate a transitive group whose degree is an arbitrary multiple of some finite number.

Dr. Stecker's first paper is in continuation of a paper by the same author under nearly the same title in the *Transactions*, volume 2, p. 152. It is shown that the problem depends upon the integration of the system of partial differential equations

$$(1)\quad \begin{aligned} EF_u - \tfrac{1}{2}EE_v - \tfrac{1}{2}FE_u &= \frac{a_1}{a_5}\Delta, \\ -GF_v + \tfrac{1}{2}GG_u + \tfrac{1}{2}FG_v &= \frac{a_4}{a_5}\Delta, \\ EG_u - \tfrac{3}{2}FE_v - \tfrac{1}{2}GE_u + FF_u &= \frac{a_2}{a_5}\Delta, \\ -GE_v + \tfrac{3}{2}FG_u + \tfrac{1}{2}EG_v - FF_v &= \frac{a_3}{a_5}\Delta, \end{aligned} \qquad \left(E_u = \frac{\delta E}{\delta u}, \text{ etc.} \right)$$

where a_i has the same meaning as in the former paper and $\Delta = EG - F^2$. These lead to two relations

$$\frac{\Delta}{\varphi(v)E^{\frac{3}{2}}} = e^{\int \frac{a2}{a_5} du} \quad e^{-3\int \frac{a_1}{a_5} \frac{F}{E} du},$$

$$\frac{G^{\frac{3}{2}}}{\varphi(u)\Delta} = e^{\int \frac{a_3}{a_5} dv} \quad e^{-3\int \frac{a_4}{a_5} \frac{F}{G} dv}.$$

The paper considers the integration of system (1) for $a_1 = a_4 = 0$ and the conclusion is that there are two solutions corresponding to one real and one imaginary surface respectively; that a real solution requires that a_2 and a_4 also vanish.

The problem of determining the number of the differential invariants of quadratic differential forms has been attacked by Zorawski and Levi-Civita. Both of these investigators used the methods of Lie's theory of continuous groups. Zorawski, however, restricted his investigations to the binary forms, and Levi-Civita determined, not the precise number of the invariants, but a lower limit for that number. In Dr. Haskins's paper the determination of the exact number of invariants of the general form in n variables is carried out. The methods used are those of Lie's theory. The work proceeds mainly by mathematical induction, and in it much use is made of certain special forms which at some stages of the process can be used instead of the general form. The four-index symbols of Christoffel also play an important part in the demonstrations. The results are that for

$$\left.\begin{matrix} n \gtreqless 3, \\ \mu > 2, \end{matrix}\right\} \quad \text{or} \quad \left\{\begin{matrix} n = 2, \\ \mu \gtreqless 4, \end{matrix}\right.$$

there are

$$I_{n\mu} \equiv \frac{n(\mu-1)}{2} \; \frac{(n+\mu-1)!}{(n-2)!\,(\mu+1)!}$$

invariants of order μ for the form in n variables.

For $n \gtreqless 3$, $\mu = 2$ there are

$$J_{n2} \equiv \frac{(n-2)(n-1)n(n+3}{12}$$

invariants, and in the cases of the invariants of lowest orders for the binary forms, viz., $n = 2$, $\mu = 2$ and $\mu = 3$, there is a single invariant. The results for $n = 2$ agree with those of Zorawski. The methods used are, however, less complicated than his.

The methods employed in this paper can be applied to the determination of the number of simultaneous invariants of several forms. The results are that for m forms the number of simultaneous invariants of order μ, exclusive of the invariants of the separate forms, is, if $\mu = 0$,

$$\frac{mn(n+1)}{2} - n^2;$$

if $n = 2$, $\mu = 3$, or if $\mu = 2$, $n > 2$, it is

$$\frac{(m-1)n(n+\mu)!}{(\mu+1)!(n-1)!} + \frac{mn(n-1)}{2};$$

and in all other cases it is

$$I_{mn\mu} \equiv \frac{(m-1)n(n+\mu)!}{(\mu+1)!(n-1)!}.$$

In these last cases the invariants may be taken as simultaneous invariants of $m - 1$ pairs of forms.

Forms containing two sets of binary variables $x_1 : x_2$ and $y_1 : y_2$ have been studied from two points of view. In what may be termed the cogredient theory, the variables are supposed to undergo the same linear transformation, so that the group defining concomitants is three-parametric

$$G_3: \quad \begin{aligned} x_1 : x_2 &= r_1x_1' + r_2x_2' : r_3x_1' + r_4x_2', \\ y_1 : y_2 &= r_1y_1' + r_2y_2' : r_3y_1' + r_4y_2'. \end{aligned}$$

In the digredient theory, on the other hand, the variables are supposed to undergo independent linear transformations, so that the fundamental group is six-parametric

$$G_6: \quad \begin{aligned} x_1 : x_2 &= r_1x_1' + r_2x_2' : r_3x_1' + r_4x_2', \\ y_1 : y_2 &= s_1y_1' + s_2y_2' : s_3y_1' + s_4y_2'. \end{aligned}$$

While the digredient concomitants of any forms are necessarily also cogredient concomitants of the same forms, the converse does not hold. After remarking that G_3 is a subgroup of G_6, consisting of those transformations of the latter which transform the bilinear form

$$\sigma \equiv x_1y_2 - x_2y_1$$

into itself, Dr. Kasner establishes the following relation between the two theories: The cogredient concomitants of any system S of double binary forms coincide with the digredient concomitants of the enlarged system S, σ obtained by adjoining σ to the original system. This is then generalized so as to apply to the group transforming an arbitrary bilinear form into itself. The results are interpreted in the theory of correspondences, in the geometry of reciprocal radius vectors, and in the geometry on the quadric surface. Special applications are made to systems of bilinear forms and to the quadri-quadric form. In the latter case use is made of the connection between double binary and quaternary forms.

In his note in the 23d volume of the *Acta Mathematica*, Mittag-Leffler has greatly extended the domain in which a function can be represented by a single analytic expression. But his results are restricted to uniform functions or to one branch of a multiform function, since the independent variable is restrained from turning about singular points. Mr. Radelfinger's paper concerns itself with removing this latter restriction and constructing an expression uniformly convergent within a multiform domain surrounding an isolated singular point, a branch point, and containing no other singular point. The method employed is founded on the notion of analytic prolongation and is briefly as follows:

A branch point of a function $F^{(z)}$ is chosen as origin 0 and a small circumference r_0 drawn around it passing through all the leaves connected at 0. A non-crossing curve C passing between 0 and all other singular points is next drawn. The curve is constructed to enclose an area of maximum extent and traverses all the leaves. It may be closed or not. The region included between r_0 and C is denoted by S. A fixed point a is then located in the region S and any other point Z taken. A curve C is drawn connecting a and Z and lying wholly within S. It is then shown that for a chosen unique initial value $F_k(a)$ the function $F_k(z)$ remains uniform within S. A length ρ is then chosen equal to the minimum distance of any point C from the boundary of S and the arc az divided into n parts locating $n-1$ intermediate points ξ such that

$$|z-\xi_{n-1}| = |\xi_{n-1}-\xi_{n-2}| = \cdots |\xi_1 - a| < \alpha^n\rho \quad (0<\alpha<1).$$

Beginning with ξ_{n-1}, circles are described about the points ξ and the point a, the radii of the circles decreasing uniformly from ρ which surrounds ξ_{n-1} to $\alpha^n\rho$ which surrounds a. Starting with the Taylor's expansion for $F(z)$ in the domain ξ_{n-1}, the function is prolonged backward along c, and the successive derivatives $F^{\lambda}(\xi_m)$ are expressed in terms of the derivatives $\overset{\lambda_1+\lambda_2}{F}(\xi_{m-1})$, until a is reached. A series n times infinite is thereby obtained which is proved to converge uniformly for all points within S for α less than one and n sufficiently large. The complete demonstration is not given but it is carried up to a point beyond which Mittag-Leffler's analysis applies without change. An interesting simplified form of the expression is obtained for the case where a circle R can be substituted for the curve C without excluding Z from the region S. To deduce this, a circle of radius r is substituted for C, and a taken as (r, o). By choosing Z sufficiently near the origin, so that ρ can be taken equal to αr, it is shown that in order to make a circuit of the origin n must be taken greater than 6.

In Dr. Snyder's paper those ruled surfaces are studied which can be generated by the line of intersection of two developables, the sum of whose classes is six. The double curve is in general of order 10, but this may be composite: only those forms are considered in which the components are straight lines and a non-reducible curve. Twenty-eight types are found.

In Dr. Stecker's second paper the broken curves C_n, of which the curve is the limit, are given for the three dimensional Hilbert curve and it is shown by considering secant planes that the osculating plane to curves of this general type are everywhere indeterminate.

Dr. Stecker's third paper is in continuation of a paper by the author in the *American Journal of Mathematics*, volume 22, p. 31, and is concerned with the properties which plane cubics and their polars have in the hyperbolic and elliptic geometries. The paper will appear in the *American Journal.*

The object of several recent papers by Professor Dickson is to show that various branches of group theory may be correlated by a treatment of groups of transformations in a given domain of rationality. The first of these memoirs, to appear in the *Transactions* for October, 1901, presents the fundamental conceptions of the theory and sets up four infinite systems of groups of transformations which are simply groups in every domain of rationality. For the case of the field of all complex numbers, these groups are the simple continuous groups of Sophus Lie. Aside from the isolated groups, the latter give the only systems of simple continuous groups of a finite number of parameters. Corresponding to the isolated group of 14 parameters, there exists in the Galois field of order p^n a (new) system of simple groups of order $p^{6n}(p^{6n}-1)(p^{2n}-1)$.

The second memoir, "Linear groups in an infinite field," communicated to the London Mathematical Society, June 13, 1901, investigates the linear group in an arbitrary field F which leaves invariant

$$\xi_0^2 - \nu\eta_0^2 + \xi_1\eta_1 + \xi_2\eta_2 + \cdots + \xi_m\eta_m,$$

where ν is not square in F, thus obtaining new simple groups. Let Q be the field obtained by the extension of F by a root of $x^2 = \nu$ and let $\bar{y}$ denote the conjugate of y with respect to F. The linear transformations on $\xi_i, \eta_i (i = 1, \cdots, m)$, with coefficients in Q of determinant unity, which leave invariant

$$\sum_{i=1}^{m} (\xi_i\bar{\eta}_i - \bar{\xi}_i\eta_i)$$

form the hyperabelian group. Its maximal invariant subgroup is formed of the transformations

$$\xi_i' = \mu\xi_i, \quad \eta_i' = \mu\eta_i \quad [\mu\bar{\mu} = 1, \ \mu^{2m} = 1] \quad (i = 1, \cdots, m).$$

Hence the group of linear fractional hyperabelian transformations of determinant unity is simple in every field.

The third memoir, offered July 18, 1901, to the *Quarterly Journal of Mathematics*, investigates the group in an arbitrary field which corresponds to the isolated simple continuous group of 78 parameters.

Mr. Rietz's paper is in abstract as follows: Let G be any non-regular transitive group of order g and degree n. Let G_1 be the subgroup of G containing all the substitutions leaving a given letter fixed. It is proved that G then contains $> \frac{g}{x+m}$ substitutions of degree $< n$, where x is the number of transitive constituents of G_1 and m the number of elements of G not occurring in G_1. When G is primitive, $m = 1$, so that a primitive group of degree n and of order g contains more than $\frac{g}{x+1}$ substitutions of degree $< n$. In particular, in a multiply transitive group of degree n more than one-half the substitutions are of degree $< n$.

From the above the following theorem is readily obtained: If G is a primitive group of degree kp (p a prime) and of order mp (m prime to p and $p-1$) the subgroup G_1 has at least $p+1$ systems of intransitivity.

In a primitive group G of odd order, G_1 is intransitive. Some theorems are given restricting the order of G_1 when it has a transitive constituent of given form. Thus, if G_1 has a transitive constituent of prime degree p, its order is not divisible by p^2.

Nearly all the theorems given have some bearing on the problem of determining all the primitive groups of odd order of a given degree. The results of a determination of all these groups of degree < 150 show that, aside from invariant subgroups of the metacyclic groups, there are only 8 such groups of degree < 150. Their orders are the following numbers: 25.3, 27.13, 27.39, 81.5, 121.3, 121.15, 125.31, 125.31.3. The first factor in these numbers as they are written gives the degree of the group of that order. Each of these groups is solvable.

In a paper by Timerding (*Crelle*, volume 123) it is shown that plane quintics with four double points may be generated by the intersection of corresponding elements of two projectively related forms, one being a linear pencil of conics and the other a quadratic pencil, viz., the system of tangents to a conic.

This theorem is easily proven in various ways and can be generalized to include the generation of any curve having four double points.

In a paper read before the Toronto Meeting of the Society, summer of 1897, Professor Dowling showed that the equation of any plane curve of order n may be put into the form

$$\alpha\beta U + \varphi^2 \Pi = 0$$

in a variety of ways, where α and β are linear expressions in the variables, U is an expression of order $n-2$, φ is a quadratic, and Π consists of the continued product of $n-4$ linear factors.

This curve will have four double points if U is made to pass through the four intersections of $\alpha\beta = 0$ and $\varphi = 0$. U will then be of the form $U \equiv \alpha\beta P + 2\varphi Q$, where P and Q are each expressions of order $n-4$. The curve will then have for its equation $\alpha^2\beta^2 + 2\varphi Q\alpha\beta + \varphi^2 \Pi = 0$ and is thus evidently generated by the two projectively related forms

$$\lambda^2 P + 2\lambda Q + \Pi = 0$$

and

$$\alpha\beta - \lambda\varphi = 0.$$

The first of the forms is a quadratic pencil of curves of order $n-4$ characterized by being tangent to the curve

$$P\Pi - Q^2 = 0.$$

This latter curve is of order $2n-8$ and possesses $n-4$ multiple tangents each touching the curve in $n-4$ points.

In Professor Dickson's second paper it is shown that the cubic form C in 27 variables with 45 terms, which was given by Cartan in his thesis on simple continuous groups, is suitable to define the configuration of the 45 triangles formed by the 27 straight lines on a general cubic surface.

In a memoir sent to the *Quarterly Journal of Mathematics*, July 18, 1901, the author studies the linear group in an arbitrary field which leaves invariant the cubic form C. For the case of a continuous field, the results agree with those of Cartan on the simple continuous group of 78 parameters. For the Galois field of order p^n, the corresponding group has the order

$$p^{36n}(p^{12n}-1)(p^{9n}-1)(p^{8n}-1)(p^{6n}-1)(p^{5n}-1)$$
$$(p^n-1)(p^n+2).$$

It may be represented as a transitive substitution group on $(p^{9n}-1)(p^{8n}+p^{4n}+1)$ letters.

The present paper also investigates the group of a quartic form Q in 56 variables with 630 terms. The partial derivative of Q with respect to any one of the variables is a cubic form differing from the above form C only in notation. The function Q defines the configuration of the 56 points of contact of the 28 bitangents to a quartic curve without double points.

In Professor Moore's second paper, from the second mean value theorem in the usual form

$$\int_a^b f(x)\varphi(x)dx = \alpha\int_a^{\xi}\varphi(x)dx + \beta\int_{\xi}^{b}\varphi(x)dx,$$
$$(\alpha = f(a) \leqq f(b) = \beta, \qquad a \leqq \xi = b),$$

where on the interval ab $f(x)$ is monotonic not decreasing and $\varphi(x)$ is integrable, is first deduced the theorem in the more general form

$$(\alpha \leqq f(a+0) \leqq f(b-0) \leqq \beta, \qquad a \leqq \xi \leqq b)$$

stated by du Bois-Reymond (*Zeitschrift für Mathematik und Physik*, volume 20 (1875), p. 126) in his review of Thomae's book on definite integrals. Two simple deductions are given, the first being doubtless that which du Bois-Reymond had in mind. The general theorem is not well known, the only preceding proof being that of Pringsheim (*Münchener Berichte*, 1900) based on a suitable generalization of a proof of the simpler theorem.

The general theorem is then extended to the case of the general improper integrals defined in the author's first paper, by a generalization of Jordan's theorem (Cours d'analyse, 2d edition, volume 2 (1894), p. 223) of the existence of the integral on the left, and by a generalization of the method used by Harnack in proving the simpler theorem for the case of his improper integrals. A corollary of this theorem is the generalization of the modified first mean value theorem (du Bois-Reymond, *Mathematische Annalen*, volume 7 (1874), p. 605).

Since the times of Euler and Lagrange it is well known that the roots of an algebraic equation of the nth degree with one unknown, may be assumed to be linear functions of n variables (or $n - 1$ variables, in case of an equation freed of the second term), in which the coefficients are different nth roots of unity. Dr. Rabinovitch in his paper remarks that these forms might yield an elegant solution in case of the quadratic, cubic, and quartic, on equating the elementary symmetric functions of these expressions of the roots to the corresponding coefficients. Thus in the case of the quartic, for instance, with second term eliminated, assuming the roots to be of the form

$$x_1 = y + z + t, \qquad x_2 = y - z - t,$$
$$x_3 = -y + z - t, \qquad x_4 = -y - z + t,$$

which is obtained from the form referred to above by putting

$$y + t = y' + t', \quad y - t = i(y' - t), \quad z = z',$$

we find their sum $= 0$, the sum of their products by twos $= -2(y^2 + z^2 + t^2)$, which is equated to the coefficient of x^2, and so on. Then he shows that the same result is obtained almost automatically, with a minimum of calculation, by equating the given equation to a circulant of corresponding order in which the leading or diagonal constituent x is the unknown of the given equation and the remaining constituents are the facients in the linear expressions of the roots.

Each of the given equations then becomes identical with a given circulant which represents the product of all the linear factors of the given equation. He establishes the generality of this procedure (with the exception, of course, of the possibility of determining the values of these facients by radicals from the obtained relations) by a theorem of Glaisher, according to which

$$C(a_0 a_1 a_2 \cdots a_{n-1}) = \prod_{r=0}^{r=n-1} (a_0 + a_1\omega_r + a_2\omega_r^2 + \cdots + a_{n-1}\omega_r^{n-1}).$$

(See Muir, Theory of Determinants § 149.)

In the case of the general quintic the same procedure yields in an easy manner a set of equations very similar to to the equations which McClintock, in his paper "Analysis of the quintic equation" (*American Journal of Mathematics*, volume 8), gives under the name of the Eulerian equations The first set may be obtained also, as indicated above, from the elementary symmetric functions of the linear expression for the roots. This last procedure, however, is very lengthy and laborious.

Dr. Black's paper, which will appear in the *Proceedings of the American Academy of Arts and Sciences*, volume 37, is in abstract as follows: Let $F(x, y, z) = 0$ be the equation of a surface, F being a function of the three independent variables x, y, z analytic in the point $x = a$, $y = b$, $z = c$; and suppose that the function and all its first partial derivatives vanish in this point. Then it is possible to represent the coördinates of all points of the surface $F = 0$ that lie in the neighborhood of the point (a, b, c) by means of a finite number of sets of parametric formulas, each of the type

$$x = f(u, v), \quad y = \varphi(u, v), \quad z = \psi(u, v),$$

where f, φ, ψ are functions of the parameters u, v, analytic each in the point $u = u_0$, $v = v_0$. A proof of this theorem was attempted by G. Kobb, *Journal de mathématiques*, series 4, volume 8 (1892), but his analysis is incomplete in essential points.

By means of a quadratic transformation of the type

$$x - a = x'(z - c), \quad y - b = y'(z - c) \tag{1}$$

the neighborhood of the point (a, b, c), in general singular, is mapped upon the neighborhood of a curve upon a new surface. If this curve does not contain multiple factors, its neighborhood, except for a finite number of neighborhoods of singular points first determined, is represented by a finite number of formulas, in each of which one coördinate is expressed as an analytic function of the other two. If the curve does, however, contain multiple factors the introduction of a quadratic transformation of the type

$$x = x'z \tag{2}$$

(preceded by a certain other transformation of the form $x + py = x_1$) secures the same result as in the other case.

Each of the new singular points is then subjected to the same treatment as the original point, and the process is repeated until all the singular points disappear, the parametric representation then becoming at once possible. It is shown that this result will be reached by means of a finite number of transformations of types (1) and (2), together with certain other one to one transformations.

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THE ITHACA COLLOQUIUM.

THE Third Colloquium of the AMERICAN MATHEMATICAL SOCIETY was held at Cornell University, Ithaca, N. Y., beginning on Wednesday, August 21, 1901, and extending over the following three days. Before describing the proceedings, it may be of interest to recall the work of the previous colloquia.

The first colloquium * organized by the Society was held in connection with its third summer meeting at Buffalo, N. Y., September 2–5, 1896. Two courses of six lectures each were delivered before an audience of thirteen members. Professor Maxime Bôcher discussed "Linear differential equations and their applications," and Professor James Pierpont, "The Galois theory of equations." The innovation proved so successful that the participants recommended to the Council that the same plan should be adopted the following summer; but the meeting at Toronto, with

* See the report, including abstracts of the courses of lectures, by Professor T. S. Fiske in BULLETIN, volume 3, pp. 49–59. Professor Bôcher's lectures were in part reproduced in the *Annals of Mathematics*, 1st series, vol. 12 (1898), pp. 45–53. Professor Pierpont's lectures were published in the same journal, 2d series, vol. 1 (1899), pp. 113–143, and vol. 2 (1900), pp. 22–56.