NON-EUCLIDEAN GEOMETRY.

Non-Euclidean Geometry. By Henry Parker Manning, Ph.D. Boston, Ginn & Co., 1901. 16mo., pp. 95.

This work is, as far as we know, the first original book upon this subject that has been published in English. Perhaps, the adjective "original" is misleading, for the author, in his preface, disclaims any attempt at originality. His object is to collect a number of the simpler theorems of non-euclidean geometry, and present them in compact and logical form to readers of slight mathematical knowledge.

At the outset, he avoids a dangerous pit-fall: metaphysics. To pass over in silence all of those philosophical questions which lie at the base of geometry, would be unpardonable in a book of more ambitious nature, but in this case, the omission seems wise. A beginner can extract but little profit from discussions of the foundations of geometry. He will accept with equal gladness Russell's contention that the conception of geometrical equality depends upon that of rigid motion, or Veronese's view that rigid motion presupposes a continuous succession of geometrically equal figures. At the same time our author makes a mistake in throwing upon the elementary text books the responsibility for most of the fundamental definitions and assumptions. We are left in doubt, for instance, whether he considers a line as the path of a moving particle or the boundary of a surface. It seems also a mistake to lay down, as universally valid, the axiom that two straight lines can meet but once, for spherical geometry is thus excluded. It is safe to say that spherical geometry is of quite as much importance as elliptic, and the study of such a figure as the sphere with two centers, might well prove attractive to beginners.* A wiser plan would be to put this axiom among the assumptions for restricted figures, and then prove that if space may be moved as a whole, two straight lines can not intersect more than twice.†

The general arrangement of the book seems to us excellent. The author first brings out the points of similarity of his three kinds of geometry, then develops their individual peculiarities. Moreover, the analytic work is placed at the end, out of consideration for those readers who prefer to avoid trigonometry, and are stampeded by the calculus.

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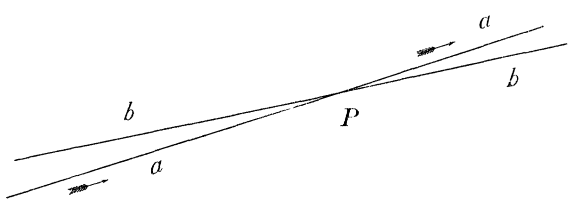
* Conf. Veronese, Grundzüge der Geometrie von mehreren Dimensionen (German translation by Schepp), p. 503.

† Killing, Grundlagen der Geometrie, p. 56.

The first chapter deals with "Pure Geometry" and gives a number of propositions for both unlimited and restricted figures, which are universally valid under the axioms laid down. Then the three hypotheses concerning the fourth angle of a trirectangular quadrilateral are stated and carried along side by side to the end of the chapter.

The second chapter deals with "Hyperbolic Geometry" and develops the theory of intersecting, parallel, and non-intersecting lines in a most elementary way, with a short notice of boundary and equidistant curves.

The third chapter, which consists of five pages only, is devoted to "Elliptic Geometry." Here it must be noticed that the author passes over in absolute silence the most striking peculiarity of the elliptic plane; namely that it is a unilateral or double surface. Now not only is this a matter of great importance in itself, but there lurks here a very serious difficulty for the beginner, which may be stated as follows: Suppose that in the elliptic plane we have two straight lines a and b lying close to one another throughout their whole extent.



Let a man start from their intersection P and make a complete circuit of the line a. If he starts in the right direction he will notice that b will at first lie on his right hand; yet when he has nearly completed the circuit and is approaching P again, b will seem to be on his left, and the change will have occurred without the two lines crossing in the meanwhile. Of course the explanation is that when he draws near to P at the end of the journey his head is pointing in the opposite direction from what it was before; but it is scarcely fair to assume that a beginner will be sharp enough to think this out.

The fourth chapter gives the analytic development of what has gone before and the book closes with a very short historical note.

The merits of the book are, then, a judicious choice of subject matter, and logical arrangement. Also the author has the rare gift of being interesting. The defects consist in carelessness in matters of detail, and occasional lack of rigor in demonstration. For instance, the whole analytic work is based on the connection between plane and spherical triangles. In Chapter I, part 3, theorem 11, we find the statement that spherical trigonometry is the same under all three hypotheses, and the reason given is that the relations of spherical figures depend merely upon the magnitude of plane and diedral angles. But the connection between a spherical and a diedral angle comes through the plane angle of the latter; what right have we to assume that two plane angles of the same diedral angle are equal? The proof given in all of the elementary books with which we are familiar is based on parallel lines, and hence is inadmissible in the present case. Incidentally, all propositions concerning mutually perpendicular planes stand or fall with the equality of these plane angles.

Fortunately, it is true that in non-euclidean geometry two plane angles of the same diedral angle are equal. An easy proof may be found from the projective definition of the magnitude of an angle with regard to the Absolute. Or a proof more suited to the present work might be developed on the following lines: The magnitude of a diedral angle may be shown to vary directly with the plane angle at any point. Hence two plane angles of the same diedral angle have a constant ratio. When one plane angle is 0, the other is an arbitrary multiple of 2π which we are at liberty to assume is 0 also. Then when one angle is π, the other is π also. Hence the factor of proportionality is 1, or the two angles are always equal.

The treatment of continuity is obscure. Axiom 4 states that "A geometrical magnitude, for example an angle, or the length of a portion of a line, varying from one value to another, passes through all intermediate values." This seems to us almost romantically vague. Moreover, unless we embody the axiom itself in our definition of the word "vary," it is not true. For instance, if we mean by an angle of a triangle an internal angle, as is habitual in elementary books, and if the base of a triangle is fixed, while the opposite vertex describes a circumference upon the base as a chord, the angle will vary from a given value to its supplement, without passing through any intermediate value. Another example of carelessness in treating continuity occurs on page 13 where we find the statement that if a line move so as always to cut off equal distance from the feet of two perpendiculars to a given line, the angles which it makes with these perpendiculars vary continuously. This is of fundamental importance, as hereon rests the proof that the same quadrilateral-hypothesis must hold everywhere in the same plane, but it should not be left without proof, merely because it is plausible, or because we ardently wish it to be true.

In the historical note at the end are serious errors. The author states (p. 92) that Saccheri, after developing elliptic, hyperbolic, and parabolic geometries side by side, states dogmatically that the two first are false. Now this is not only incorrect, but it conveys an unfair impression of the Jesuit mathematician. He does not state dogmatically that these geometries are false, but gives an elaborate and ingenious sequence of propositions in each case, to show that the system is self-contradictory.* Again the author states on the following page, that elliptic geometry was discovered by Riemann. This seems to us rash. It is by no means clear whether Riemann looked upon the geodesics of a surface of constant positive curvature as cutting in one or two points: the probability being that he held the latter view, so that it is customary to credit Klein and Newcomb with the discovery of elliptic geometry.†

We may say, in conclusion, that the book is a contribution to educational rather than to mathematical literature. It is neither a scholarly nor a profound work, but comes in answer to a real need, and marks a step in advance.

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BIANCHI'S DIFFERENTIAL GEOMETRY.

Vorlesungen über Differentialgeometrie. Von LUIGI BIANCHI. Autorisierte deutsche Uebersetzung von MAX LUKAT. Leipzig, B. G. Teubner, 1896–1899. Pp. xvi + 659.

IN 1886 Bianchi published a lithographed edition of his lectures on differential geometry given at the University of Pisa during the winter 1885–86. This publication, on which the book now before us is based, consisted of only fourteen chapters. The *Vorlesungen* contains twenty-two. Professor Loria, of Genoa, says of the lithographed edition :‡

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* Conf. Engel und Staeckel, Die Theorie der Parallellinien von Euklid bis auf Gauss, pp. 67 and 109.

† Killing, loc. cit., p. 70. Russel, The Foundations of Geometry, pp. 39, 40.

‡ *Jahrbuch über die Fortschritte der Mathematik*, 1886, p. 648.