NOTE ON HAMILTON'S DETERMINATION OF IRRATIONAL NUMBERS.

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THE purpose of this note is to call attention to Hamilton's use of the partition (Schnitt) in his definition of certain irrational numbers.*

Though Hamilton's method of discussing the irrational number is open to serious criticism, it has some interest as affording an approximation to the rigorous theory afterward developed by Dedekind†, and also as presenting another example of the natural character of Dedekind's definition of the irrational number. ‡

For Hamilton time rather than space furnished the rudiment to which his intuition appealed for heuristic purposes and for his conception of unity. Starting with this conception he gives a careful exposition of the operations on positive and negative integers which do not carry one outside the system. A similar discussion of the rational number follows. From the analogy furnished by the laws of operations on integral and rational numbers, Hamilton writes expressions by which he defines the fundamental operations on ratios of any two intervals (steps) of time, and assumes that to any such ratio corresponds a number which he calls an "algebraic number." This he does, even though he has considered the existence of no numbers save integers and fractions In proceeding to develop his theory of irrational number Hamilton too often makes use of these "algebraic numbers," of whose existence and properties he has proved nothing. Throughout his work, however, fractions could be substituted for "algebraic numbers" and such a change would go far toward making his work rigorous In the following outline of Hamilton's work, it is assumed unless the

* "Theory of conjugate functions, or algebraic couples; with a preliminary and elementary essay on algebra as the science of pure time," *Transactions of the Royal Irish Academy*, vol. 17 (1837), p. 293. This essay was read in 1833.

† Stetigkeit und irrationale Zahlen, 1872

‡ In commenting on Tannery's development of the partition (Introduction à la théorie des fonctions d'une variable, 1886) from a remark made by Bertrand (Traité d arithmétique, p. 203, 1849) Dedekind says, (Was sind und was sollen die Zahlen, 1893, p. xiii): "Diese Uebereinstimmung scheint mir ein erfreulicher Beweis dafür zu sein, dass meine Auffassung der Natur der Sache entspricht."

contrary is stated, that the numbers considered are "algebraic numbers."

Hamilton first shows as a corollary of a theorem on multiplication that for any numbers x' and x''

$$x'^2 \lesseqgtr x''^2 \text{ according as } x' \lesseqgtr x''.$$

He next assumes that between any two fixed numbers an infinite number of numbers must lie. Suppose A and B be two finite sets of numbers, arranged in order of magnitude, increasing and decreasing respectively, such that the greatest number in A is less than the least number in B. There must exist in general an infinite number of numbers less the least number in B and greater than the greatest number in A. Hamilton goes on to say that if the numbers in A and B are not given explicitly, but by some law, it may happen that no rational number can exist between A and B. But his intuition indicated that such sets could be used to determine non-rational numbers. This he elaborates in detail for a few particular cases, as for example the square root of a positive fraction. His work on this case is in outline as follows :

Let b denote a positive number. Suppose we have the sets A and B composed of an infinite number of rational elements

$$\alpha_1 < \alpha_2 < \alpha_3 < \cdots \quad \text{and} \quad \beta_1 > \beta_2 > \beta_3 \cdots,$$

respectively, where

$$\alpha_i^2 < b < \beta_j^2 \qquad (i, j = 1, \cdots).$$

Hamilton shows that these two sets A and B determine uniquely a quantity which he calls the irrational number a. He defines the product a^2 by the partition (A', B') where the elements of A' and B' are

$$\alpha_1^2 < \alpha_2^2 < \alpha_3^2 < \cdots \quad \text{and} \quad \beta_1^2 > \beta_2^2 > \beta_3^2 > \cdots,$$

respectively. He then shows that the number afforded by this partition is neither greater nor less than b. Thus he calls

$$a = \sqrt{b}.$$

Hamilton does not define addition or subtraction of partitions, nor does he order them. Neither does he show that every partition of real numbers defines a real number.

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