POSSIBLE TRIPLY ASYMPTOTIC SYSTEMS OF SURFACES.

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In my note, entitled "A demonstration of the impossibility of a triply asymptotic system of surfaces,"[*] which was read before the AMERICAN MATHEMATICAL SOCIETY at its December meeting, I failed to take account of an exception which presents itself in the discussion. It was found that in order that there exist a triply asymptotic system of surfaces, it is necessary and sufficient that the double system of asymptotic lines of each surface shall at the same time be geodesic lines on the surfaces. This double condition can be satisfied only when the asymptotic lines in both systems are rectilinear generators of the surface. The quadric surfaces are known to possess this peculiar property, and for the hyperboloid of one sheet and the hyperbolic paraboloid these generatrices are real. Hence instead of the general negation previously given we have the qualified one :

The only triple systems of surfaces cutting mutually in the real asymptotic lines of these surfaces are composed of properly associated families of hyperboloids of one sheet and hyperbolic paraboloids.

One such system can be gotten as follows :

As, in the previous note, we consider space referred to any system of curvilinear coördinates ρ_1, ρ_2, ρ_3 and let the cartesian coördinates x, y, z of a point with respect to fixed rectangular axes be given in terms of ρ_1, ρ_2, ρ_3 by the equations

$$(1) \quad x = f(\rho_1, \rho_2, \rho_3), \quad y = \varphi(\rho_1, \rho_2, \rho_3), \quad z = \psi(\rho_1, \rho_2, \rho_3).$$

We have remarked that the coefficients of the system

$$(2) \quad \begin{aligned} \frac{\partial^2 \theta}{\partial \rho_1{}^2} &= a_{11} \frac{\partial \theta}{\partial \rho_1} + a_{12} \frac{\partial \theta}{\partial \rho_2} + a_{13} \frac{\partial \theta}{\partial \rho_3}, \\ \frac{\partial^2 \theta}{\partial \rho_2{}^2} &= a_{21} \frac{\partial \theta}{\partial \rho_1} + a_{22} \frac{\partial \theta}{\partial \rho_2} + a_{23} \frac{\partial \theta}{\partial \rho_3}, \\ \frac{\partial^2 \theta}{\partial \rho_3{}^2} &= a_{31} \frac{\partial \theta}{\partial \rho_1} + a_{32} \frac{\partial \theta}{\partial \rho_2} + a_{33} \frac{\partial \theta}{\partial \rho_3}, \end{aligned}$$

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can be so determined that it will admit x, y, z, as particular simultaneous solutions.   And we have found that the necessary and sufficient condition that the families of surfaces

$$\rho_1 = \text{const.}, \quad \rho_2 = \text{const.}, \quad \rho_3 = \text{const.},$$

shall cut one another in their asymptotic lines is expressed by

(3) $a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0.$

Consider now in particular the case where x, y, z are such that they satisfy the equations

(4) $\dfrac{\partial^2 \theta}{\partial \rho_1^2} = 0, \quad \dfrac{\partial^2 \theta}{\partial \rho_2^2} = 0, \quad \dfrac{\partial^2 \theta}{\partial \rho_3^2} = 0.$

In order that this system may be consistent with system (2), it is necessary that, either all the coefficients in the right hand members of the latter be zero, or that x, y, z, as functions of ρ_1, ρ_2, ρ_3, satisfy the condition

$$\frac{\partial(x,\ y,\ z)}{\partial(\rho_1,\ \rho_2,\ \rho_3)} = 0,$$

i. e., that there exist a relation between x, y, z.   As this is impossible, we must have

(5) $a_{11} = a_{12} = \cdots = a_{33} = 0.$

It is evident that the most general expressions which x, y, z can have under the conditions (4) are

$$x = a_1\rho_1\rho_2 + b_1\rho_1\rho_3 + c_1\rho_2\rho_3 + d_1\rho_1 + e_1\rho_2 + f_1\rho_3 + g_1,$$
(6) $$y = a_2\rho_1\rho_2 + b_2\rho_1\rho_3 + c_2\rho_2\rho_3 + d_2\rho_1 + e_2\rho_2 + f_2\rho_3 + g_2,$$
$$z = a_3\rho_1\rho_2 + b_3\rho_1\rho_3 + c_3\rho_2\rho_3 + d_3\rho_1 + e_3\rho_2 + f_3\rho_3 + g_3,$$

where a_1, b_1, $\cdots$, g_3 are arbitrary constants.   The cartesian coördinates of a point on one of the surfaces $\rho_3 = \text{const.}$ are given as the following expressions of the parameters ρ_1 and ρ_2, which evidently refer to the double system of lines in which the surface is cut by the two families of surfaces $\rho_1 = \text{const.}$, $\rho_2 = \text{const.}$,

$$x = a\rho_1\rho_2 + \beta\rho_1 + \gamma\rho_2 + \delta,$$
(7) $$y = a_1\rho_1\rho_2 + \beta_1\rho_1 + \gamma_1\rho_2 + \delta_1,$$
$$z = a_2\rho_1\rho_2 + \beta_2\rho_1 + \gamma_2\rho_2 + \delta_2,$$

where the expressions for a, $\cdots$, δ_2, as functions of ρ_3, a_1, $\cdots$, g_3, are readily found by comparing systems (7) and (6).

Eliminating ρ_1 and ρ_2 by means of Sylvester's dialytic method, we find the following equation of the second degree :

(8) $(a_1\beta_2)(a_1\gamma_2)x^2 + (a\beta_2)(a\gamma_2)y^2 + (a\beta_1)(a\gamma_1)z^2 - [(a_1\beta_2)(a\gamma_2)$

$$+ (a_1\gamma_2)(a\beta_2)]xy + [(a_1\beta_2)(a\gamma_1) + (a_1\gamma_2)(a\beta_1)]xz -$$
$$[(a\beta_2)(a\gamma_1) + (a\gamma_2)(a\beta_1)]yz + Ax + By + Cz + D = 0,$$

where, for the sake of brevity, we have written

$$(a_i\beta_k) \equiv a_i\beta_k - a_k\beta_i,$$

and A, B, C, D are functions of $a, \cdots, \gamma_3$, whose form is of no consequence in the discussion except that they are finite for finite values of the latter.

Denoting by $\varDelta$ the discriminant of the terms of second degree in (8), we find that

$$(9) \qquad\qquad \varDelta = 0.$$

In consequence of this and of the character of A, B, C, as remarked, the quadric whose equation is (8) has its center at an infinite distance. By giving to ρ_3 successive values in this equation, we get the equations of the form

$$f(x, y, z) = \text{const.},$$

defining the family of surfaces $\rho_3 = \text{const.}$ From (6) and (7) we see that the asymptotic lines on these surfaces are real when ρ_3 and $a_1, \cdots, g_3$ are real; hence these surfaces $\rho_3 = \text{const.}$ are of total negative curvature. Moreover, these asymptotic lines are distinct for general values of $a_1, \cdots, g_3$. Hence $\rho_3 = \text{const.}$ defines a family of hyperbolic paraboloids.

Since the formulæ (6) are perfectly symmetrical with respect to ρ_1, ρ_2, ρ_3, it follows that in a similar manner we can show that $\rho_1 = \text{const.}$ and $\rho_2 = \text{const.}$ define respective families of hyperbolic paraboloids. Hence the formulæ (6) serve to define a triply asymptotic system of hyperbolic paraboloids.

Consider the surface $\rho_3 = k_3$, where k_3 is a fixed constant. The cartesian coördinates of a point on this surface are given by (7), in which ρ_3 has been replaced by k_3. If in the expressions thus obtained we put ρ_2 equal to k_2, a fixed constant, we get the coördinates of a point whose locus is the asymptotic line in which the surfaces $\rho_3 = k_3$ and $\rho_2 = k_2$ intersect, asymptotic for both surfaces. These expressions may be written

$$(10) \qquad x = l\rho_1 + p, \quad y = m\rho_1 + q, \quad z = n\rho_1 + r,$$

where $l, m, \cdots, r$ are constant along the line; their expressions are readily found from (6). The equations (10) show that the curve along which $\rho_3 = k_3$, and $\rho_2 = k_2$ intersect is a right line—a rectilinear generator for each surface—as we have seen from other considerations.

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