

DINI'S METHOD OF SHOWING THE CONVERGENCE OF FOURIER'S SERIES AND OF OTHER ALLIED DEVELOPMENTS.

BY MR. WALTER B. FORD.

(Read before the American Mathematical Society, December 28, 1900.)

IF it is required to show that the arbitrary function  $f(x)$  of the real variable  $x$  may be developed into a Fourier series for all values of  $x$  lying between  $-\pi$  and  $\pi$  ( $-\pi$  and  $\pi$  at most excluded) then, remembering that for any particular value of  $x$ , such as  $x = a$ , the sum of the first  $n + 1$  terms of the series is

$$(1) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \frac{\sin \frac{2n+1}{2}(x-a)}{\sin \frac{1}{2}(x-a)} dx,$$

it is easily shown that we need merely examine the limits approached by the integrals

$$(2) \quad \int_0^a f(x) \frac{\sin \frac{2n+1}{2}(x-a)}{\sin \frac{1}{2}(x-a)} dx,$$

$$(3) \quad \int_c^a f(x) \frac{\sin \frac{2n+1}{2}(x-a)}{\sin \frac{1}{2}(x-a)} dx,$$

as  $n$  increases indefinitely,  $c$  and  $d$  being any numbers such that  $0 < c < d \leq \frac{1}{2}\pi$ . In case these limits exist and have certain simple properties it becomes evident that the given series will be convergent for any value  $x = a$  which lies between  $-\pi$  and  $\pi$ , and will have as its sum either  $f(a)$  or

$$\frac{f(a+0) + f(a-0)}{2},$$

while at either of the points  $x = -\pi$  or  $x = \pi$  the sum will be

$$\frac{f(-\pi+0) + f(\pi-0)}{2}$$

Moreover, a closer examination of integrals (2) and (3) shows that if  $f(x)$  satisfies any one of certain sets of preliminary conditions (of which Dirichlet's conditions are a special case), then these integrals (2) and (3) possess the required properties; for in such cases the problem reduces to the establishing of certain well known properties of the more simple integrals

$$(4) \quad \int_0^d \frac{\sin \frac{2n+1}{2}(x-a)}{\sin \frac{1}{2}(x-a)} dx,$$

$$(5) \quad \int_c^d \frac{\sin \frac{2n+1}{2}(x-a)}{\sin \frac{1}{2}(x-a)} dx,$$

where  $c$  and  $d$  have the same significance as before. In such cases, then, the original problem reduces to questions which are entirely independent of the function  $f(x)$  which was to be developed. Now, integrals (4) and (5) being once closely examined and their properties which are essential to the present problem being once tabulated, the question arises whether there are not still other functions of  $x$ ,  $n$  and  $a$ , which like

$$\frac{\sin \frac{2n+1}{2}(x-a)}{\sin \frac{1}{2}(x-a)},$$

when integrated between certain determined or undetermined limits  $(0, d')$  in the one case and  $(c', d')$  in the other, will possess the same essential properties which belong to (4) and (5), and, as thus integrated, will therefore form integrals bearing the same relation to *new* series developments of  $f(x)$  for  $x$  between certain limits  $(a, b)$  that (4) and (5) bore to the Fourier development of  $f(x)$  for  $x$  between  $-\pi$  and  $\pi$ . Evidently if such functions exist, representing any one of them by  $\varphi(x-a, a, n)$ , or better by  $\varphi(x-a, a, h_n)$  where  $h_n$  is any particular expression which (like the  $\frac{2n+1}{2}$  which appears in (4) and (5)) is always positive and increases indefinitely with  $n$ , the integrals in question will be

$$(6) \quad \int_0^{d'} \varphi(x-a, a, h_n) dx,$$

$$(7) \quad \int_{c'}^{d'} \varphi(x-a, a, h_n) dx,$$

and for any particular value of  $x$  such as  $x = a$  it is evident that the sum of the first  $n + 1$  terms of this new series will be an integral corresponding to the integral (1), and in fact it may be easily shown that this sum will be

$$(8) \quad \frac{1}{2G} \int_a^b f(x) \varphi(x - a, a, h_n) dx,$$

where  $G$  is a determinate constant independent of  $x$  and of  $a$  and different from zero.

Dini begins by showing that there are an infinite number of such functions  $\varphi(x - a, a, h_n)$ \*. Then considering the series

$$(9) \quad \frac{1}{2G} \int_a^b f(x) \varphi(x - a, a, h_0) dx \\ + \frac{1}{2G} \sum_1^{\infty} \int_a^b f(x) \{ \varphi(x - a, a, h_n) - \varphi(x - a, a, h_{n-1}) \} dx,$$

the sum of whose first  $n + 1$  terms is evidently the above expression (8), he proceeds to make farther assumptions as regards the functions  $\varphi(x - a, a, h_0)$ ,  $\varphi(x - a, a, h_1)$ , ...  $\varphi(x - a, a, h_n)$ ; *i. e.*, he assumes that each of these functions is such that every term of (9) may be expressed in a certain typical form (which is the form common to all known developments). One of the chief requirements of a term in this form is that it shall involve in a specified way certain special functions of  $x$ , or of  $x$  and  $\lambda_n$ , where  $\lambda_n$  is a parameter variable only with  $n$ , which functions we may suppose to have taken in advance as the functions in terms of which any new development might be desired (these functions in the case of the Fourier series being  $\sin nx$  and  $\cos nx$ .)

This being done, it is evident that if  $f(x)$  satisfies any one of the sets of conditions mentioned before, we may now form a sufficient condition that this function may be developed for all values of  $x$  lying between the determined or undetermined limits  $(a, b)$  ( $a$  and  $b$  at most excluded) in terms of given functions of  $x$ , or of  $x$  and  $\lambda_n$ , the development being of the form common to all known developments (developments such as occur, for example, in the general study of mathematical physics).

This condition is described by saying that the sum of the first  $n + 1$  terms of the given series should be equal to an

---

\* Reference is here made to Dini's work entitled: *Serie di Fourier e altre rappresentazioni analitiche delle funzioni di una variabile reale*; Pisa, 1880.

integral of the form (8), where  $\varphi(x - a, a, h_n)$  is a function of the type mentioned above. But to use this condition as a criterion in any case where we desire to test a given development is evidently difficult, for we are thus required to determine certain facts about the nature of an infinite series whose individual terms are generally of a complicated character. As an aid to this undertaking, Dini now employs an ingenious method due in substance to Cauchy. He virtually shows how a function of the complex variable  $z$  may be constructed so that it will have an infinite number of poles of the first order distributed at finite distances from each other along the positive side of the axis of reals, this function being otherwise monogenic within finite regions of the  $z$  plane, and being, moreover, so constructed that its residua in these poles (or at least in some of them) are equal respectively to the individual terms of the given series in  $x$  which we are to test. This once accomplished, it is evident that we have merely to integrate this function of  $z$  about any closed contour lying to the right of the axis of pure imaginaries and enclosing the first  $n + 1$  of these poles and not passing through any one of them, in order to obtain readily in the form of a definite integral (in the complex) the sum of the first  $n + 1$  terms of the given series; and by enlarging the contour so as to enclose more and more of the poles we shall have in the limit as the contour thus enlarges indefinitely an expression in the form of a definite integral (in the complex) which actually represents the sum of the given series. Thus, by this means our investigation is transferred from questions regarding the given series to questions regarding a definite integral which varies with  $n$ , and for any special value of  $n$  is equal to the sum of the first  $n + 1$  terms of the given series. Whenever this integral reduces to one of the integrals (8) we are assured that the given development is possible. To study this reduction is, however, difficult in many cases, but Dini shows how conclusive results may be obtained in the case of Fourier series, series in terms of zonal harmonics, Bessel functions, and elliptic functions, and he makes it evident that the method leads to decisive results in many other cases.

ANN ARBOUR,

December, 1900.