It follows that Γ'' is identical with $SH(6, 2^n)$. But Γ'' is holoedrically isomorphic with $G_{p=2}$ and therefore with $HA(4, 2^{2n})$, whose second compound is $G_{p=2}$.

4. NOTE.—It appears that the quaternary transformation group which naturally corresponds to the finite group $HO(4, p^{2n})$ is *not* continuous.

THE UNIVERSITY OF TEXAS,
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THE HESSIAN OF THE CUBIC SURFACE. II.

BY DR. J. I. HUTCHINSON.

(Read before the American Mathematical Society, February 24, 1900.)

THE aim of the following paper is to extend the results of a previous article on the same subject (BULLETIN, March, 1899, p. 282) by determining all the quintic and sextic curves on the Hessian of the cubic surface, and giving some theorems connected with them, and with the quartic curves already determined.

I will write the equation of the Hessian in the form

$$F \equiv xyzu + wyzu + wxzu + wxyu + wxyz = 0,$$

where w, x, y, z, u are connected by the relation

$$aw + bx + cy + dz + eu = 0,$$

in which a, b, c, d, e are arbitrary constants.

As already shown, the surface F contains three classes of biquadratic curves, viz.:

α_1. A class containing 15 families which lie on 30 families of cones, all the cones of the same family cutting F in two lines and tangent along a third.

α_2. A class containing 30 families of curves lying on 30 families of cones tangent to F along two lines.

α_3. A class containing 15 families of curves determined by as many families of quadrics each intersecting F in a gauche quadrilateral, and by 30 families of quadrics each meeting F in two lines and a conic.

Consider the family of α_1 determined by the cones

$$(1)\qquad \begin{aligned} A_1 &\equiv x(w + y) + \lambda wy = 0, \\ A_1' &\equiv x(z + u) + (1 - \lambda)zu = 0. \end{aligned}$$

The plane $w + y = 0$ is tangent to all of the cones A_1 and hence is a double tangent plane for each biquadratic curve of the family. The two points of tangency with each curve form pairs of points in an involution on the line wy. Similarly, the plane $z + u = 0$ is a double tangent plane for each curve of the family, the points of contact forming an involution on the line zu.

All the curves of the family pass through the four nodes of F which lie on the lines wx and xy excluding their point of intersection (or we may say, the four which lie in the plane x excluding the vertices of A_1 and A_1'). These are the only points common to two curves of the same family.

The 15 families of α_1 can be grouped in 5 sets of 3 each, any one set having the property that a cubic surface can be passed through three curves arbitrarily chosen, one from each family of the set. For example, with the family determined by equations (1) are associated the two others obtained by the permutations (wz) and (wu). The family of cubic surfaces which intersects F in these three families of curves has the equation

$$A_1[x + \lambda'(1 - \lambda'')z + \lambda''(1 - \lambda')u] + A_1'[(1 - \lambda')(1 - \lambda'')w + x + \lambda'\lambda''y] = 0,$$

since the left member is unaltered by the permutations $(wz)(\lambda\lambda')$ and $(wu)(\lambda\lambda'')$.

Consider the family α_3 determined by the intersection of the quadrics

$$(2) \qquad \begin{aligned} A_3 &\equiv wy + \lambda zu = 0, \\ A_3' &\equiv wx + wy + xy - \lambda x(z + u) = 0. \end{aligned}$$

Each curve of the family passes through the same four nodes of F which form the points of intersection of the curve of the family (1) of α_1. These two families of α_1 and α_3 are also associated in such a way that they form the complete intersection with F of the family of quadrics

$$A_1 - \lambda' A_1' \equiv (\lambda - 1)A_3 + A_3' = 0.$$

The four points in which a given curve of the family (2) meets the lines wz, wu, yz, yu lie in a plane whose equation is

$$y + w - \lambda(z + u) = 0.$$

The plane $w + u = 0$ intersects A_3 in two lines one of which is the line wu. The plane is accordingly tangent to

A_3 in the point whose coördinates are $w = u = 0$, $y = \lambda z$. These coördinates also satisfy the equation $A_3' = 0$. Hence, the plane $w + u$ touches each biquadratic of the family in that point of the line wu where it is tangent to the quadric A_3 containing the given curve. Similarly for the planes $w + z$, $y + z$, $y + u$.

By means of the permutations (wz) and (wu) applied to the equations (2) we determine two other families which together with the first are so related that a cubic surface can be passed through three curves arbitrarily chosen, one from each family. The equation of the cubic is

$$A_3[\lambda''w + (1 + \lambda')(1 + \lambda'')x + \lambda'y] - A_3'[\lambda''w + \lambda'y - \lambda'\lambda''z - u] = 0,$$

since the left member is unaltered by the permutations $(yu)(\lambda\lambda')$ and $(wu)(\lambda\lambda'')$.

The curves (1) of α_1, and (2) of α_3 have in common the property that they meet any of the conics $\{xw\}$ (in the plane $w + x = 0$), $\{xy\}$, $\{xz\}$, $\{xu\}$ in pairs of points forming an involution.

Any curve of (1) intersects any curve of (2) in 8 points, on account of the identity

$$A_1 - \mu A_1' + (1 - \lambda)A_3 - A_3' \equiv 0,$$

where A_1, A_1' contain the parameter λ, and A_3, A_3' contain μ. But since

$$(1 - \lambda)A_1 + \lambda\mu A_1' - \lambda(1 - \lambda)A_3 \equiv x[(1 - \lambda)(w + y) + \lambda\mu(z + u)],$$

it follows that the four movable points of intersection lie in the plane

$$(1 - \lambda)(w + y) + \lambda\mu(z + u) = 0,$$

the remaining four being in the plane $x = 0$ as already noticed.

Consider the family of α_2 determined by the cones

$$(3) \quad \begin{aligned} A_2 &\equiv \lambda w^2 + wx + wy + xy = 0, \\ A_2' &\equiv \lambda(zu + wz + wu) + (x + y)(z + u) = 0. \end{aligned}$$

Two curves of the same family do not intersect.

The planes $w + x$, $w + y$, and $z + u$ are double tangent planes to each curve of the family, the pairs of points so determined forming involutions on the lines wx, wy, and zu respectively.

The four lines xz, yz, xu, yu are met by each curve in one point each, the four points so determined lying in the plane

$$\lambda w + x + y = 0.$$

If we apply to equations (3) the permutations (wxz) and (wzx), we obtain two other families of curves which together with (3) form the complete intersections of F and a family of cubic surfaces whose equation is

$$A_2[\lambda' x + y + \lambda''(1-\lambda')z - (1-\lambda')(1-\lambda'')u] \\ + A_2'[(\lambda'\lambda'' - \lambda'' + 1)w + \lambda'\lambda'' x + y + \lambda'' z] = 0.$$

The function in the left member is unchanged by the permutation $(wxz)(\lambda\lambda'\lambda'')$. By means of the permutations (xy), (zu), and $(xy)(zu)$ which leave A_2 and A_2' unchanged we see that the family (3) is also associated in a similar manner with three other pairs of families.

There are four classes of quintic curves, all of the first species according to Salmon's classification. (Geometry of three dimensions, 4th edition, p. 318.)

β_1. Ten families of quintics formed by ten families of quadric cones passing through three lines of F which meet in a point. Such are the cones

$$(4) \qquad B_1 \equiv \mu_1 xy + \mu_2 wy + \mu_3 wx = 0.$$

The same family of quintics is also determined by the partial intersection of F and the three families of cubic surfaces

$$\Gamma_1 \equiv \mu_1 xy(z+u) + zu[(\mu_1-\mu_3)x + (\mu_1-\mu_2)y] = 0,$$
$$\Gamma_2 \equiv \mu_2 wy(z+u) + zu[(\mu_2-\mu_3)w + (\mu_2-\mu_1)y] = 0,$$
$$\Gamma_3 \equiv \mu_3 wx(z+u) + zu[(\mu_3-\mu_2)w + (\mu_3-\mu_1)x] = 0,$$

since

$$\mu_1 F \equiv w\Gamma_1 + zuB_1, \quad \mu_2 F \equiv x\Gamma_2 + zuB_1, \quad \mu_3 F \equiv y\Gamma_3 + zuB_1.$$

If Γ_i' denote the function Γ_i with μ_k' written in the place of μ_k, it is easy to see that the two curves whose parameters are μ_k and μ_k' lie on the cubic surface Γ, where

$$\Gamma \equiv \mu_1'\Gamma_1 + \mu_2'\Gamma_2 + \mu_3'\Gamma_3 \equiv \mu_1\Gamma_1' + \mu_2\Gamma_2' + \mu_3\Gamma_3'.$$

For $\mu_k' = \mu_k$ we obtain a family of cubic surfaces inscribed to F along the quintics of the family (4) and the line zu.

By taking $\mu_i = 0$ we see that this family of quintics includes three families of plane cubics, and by taking $\mu_i = \mu_k$ we find that it also includes three families of biquadratics of α_1.

Every curve of the family passes through the seven nodes lying on the lines wx, wy, and xy. Any two curves intersect in two other points whose joining line is evidently a common generator of the two cones on which the two curves lie.

Through any two curves of β_1, whether belonging to the same family or not, a cubic surface can be passed which meets F elsewhere in two lines.

β_2. The next class of quintics contains sixty families determined by sixty families of cones, each passing through intersecting lines of F and tangent along one of them. The equation

$$(5) \qquad B_2 \equiv \alpha y^2 + \beta wy + \gamma x(w + y) = 0$$

leads to a family of this class which is also determined by the cubic surfaces

$$G \equiv \alpha y(xz + xu + zu) + (\alpha - \beta + \gamma)xzu + \gamma x^2(z + u) = 0,$$

since

$$(\alpha - \beta)F \equiv G(w + y) - B_2(xz + uz + xu).$$

Every curve of the family passes through the three nodes which lie on the line xy. Any two of the curves intersect in two other points.

β_3. The third class contains sixty families determined by sixty families of quadrics passing through two non-intersecting lines of F and a third line cutting both. The two equations

$$(6) \qquad \begin{aligned} B_3 &\equiv y(\lambda w + \mu x) + wz = 0, \\ H_3 &\equiv xu(w + z) - (\lambda w + \mu x)(yu + xu + xy) = 0, \end{aligned}$$

determine such a family, since

$$F \equiv yH + B_3(yu + xu + xy).$$

Every curve of the family passes through the five nodes situated on the lines wz, wy, and yz excluding the two points of intersection. Any two curves have two other points of intersection.

If we apply to equations (6) the permutations $(wx)(zu)$, $(yz)(xu)$, $(wz)(yu)$, $(wu)(xy)$, we obtain together with (6) five families of β_3 having the property that if two curves be taken from any two different families of the set, they will lie on a cubic surface which either passes through a conic, or is tangent to F along a line. The 60 families of β_3 group themselves into 12 sets of 5 of the like character.

β_4. The fourth class contains 30 families determined by 30 families of quadrics meeting F in a line and a conic. Such a family is given by the equations

$$B_4 \equiv zu + yu + yz - (w + x)(\lambda y + \mu z) = 0,$$
$$J \equiv \lambda wxy + \mu wxz + yzu = 0,$$

since
$$F \equiv wxB_4 + (w + x)J.$$

Any two curves of the family intersect in five points, three of which are the nodes contained on the line yz.

There are six different classes of sextic curves on the Hessian.

γ_1. The first class contains 30 families, each of which is determined by the quadrics passing through two intersecting lines of F. The curves determined by the equation

$$C_1 \equiv wp + xy = 0,$$

where
$$p = \alpha w + \beta x + \gamma y + \delta z + \varepsilon u,$$

also lie on the cubic surfaces

$$K_1 \equiv - zup + xyz + xyu + xzu + yzu = 0,$$

since
$$F \equiv zu C_1 + wK_1.$$

Any two curves of this family with parameters $\alpha, \cdots$ and $\alpha_1, \cdots$ intersect in the four nodes lying on the lines wx and wy, excepting their common point, and in six other points lying on a conic in the plane $p - p_1 = 0$.

A second family generated by the equations

$$\overline{C}_1 \equiv wp' + zu = 0,$$
$$\overline{K}_1 \equiv - xyp' + xyz + xyu + xzu + yzu = 0$$

is associated with the preceding in such a way that a curve from each family lies on the cubic surface

$$K \equiv p' C_1 - K_1 \equiv p \overline{C}_1 - \overline{K}_1 = 0.$$

γ_2. The second class, containing 10 families, is determined by quadrics which touch F along a line. Such a family is

$$C_2 \equiv (w + x)p + wx = 0,$$

also determined by the surfaces

$$K_2 \equiv - yzu + p(yz + yu + zu).$$

In this case we have

$$F \equiv C_2(yz + yu + zu) - (w + x)K_2.$$

γ_3. The third class also contains 10 families. The quadric

$$C_3 \equiv (w + x)p + yz + yu + zu = 0,$$

which meets F in a conic, and the cubic surface

$$K_3 \equiv wxp - yzu = 0$$

determine such a family, since

$$F \equiv wxC_3 - (w + x)K_3.$$

Any two curves of the same family in either γ_2 or γ_3 intersect in six points lying on a conic.

A curve from each of the two families of γ_2 and γ_3 just mentioned lies on the cubic surface

$$K \equiv C_2p' + K_2 \equiv C_3' p + K_3'.$$

γ_4. The fourth class contains 15 families, each determined by the intersection with F of a family of quadrics such as

$$\alpha wy + \beta wz + \gamma xy + \delta xz = 0,$$

passing through two non-intersecting lines wx, yz. A sextic of this class cannot lie on a cubic surface.

γ_5. The fifth class of sextics contains 60 families. These curves are formed by the partial intersection of two cubic surfaces such as

$$\triangle_1 \equiv y(wx + wz + xz) + z(\alpha wx + \beta xz + \gamma wz) = 0,$$
$$\triangle_2 \equiv u(\alpha wx + \beta xz + \gamma wz) - wx(y + u),$$

which intersect in the three lines wx, wz, xz, and a sextic on F, since

$$F \equiv u\triangle_1 - z\triangle_2.$$

γ_6. The sixth class contains a single family, the Steiner sextics, determined by the equation

$$S \equiv \kappa xyz + \lambda wyz + \mu wxz + \nu wxy = 0,$$

or equally well by four other equations referred to the four other fundamental tetrahedra.

Let s and s' be any two curves of the family and let $T = 0$, $T' = 0$ be the two cubic surfaces which touch F along s and s' respectively. Also let $H = 0$ be the cubic surface containing both s and s'. Then an identity of the form

$$(\mathrm{I}) \qquad QF \equiv TT' + H^2$$

necessarily exists, where Q is a function of the second degree. A course of reasoning similar to that used by Humbert in connection with the Kummer surface enables us to draw a number of interesting conclusions from (I).

The two curves s and s' intersect in four points p_1, p_2, p_3, p_4 (besides the 10 nodes of F through which every curve of the family passes). The five surfaces Q, F, T, T', H are mutually tangent at each of these four points.

The surface T intersects H in s and a twisted cubic c. It is evident that c lies on Q and that T touches Q along this cubic. Similarly, T' touches Q along a cubic c' which also lies on H.

The 24 points of intersection of Q, H, and F are double points of the surface $TT' = 0$. Among these occur the points p_i, each counted four times. The remaining eight points, since they lie on the curves s and s', are not points of tangency of T and T'. They must therefore be nodes on either T or T'. It is evident that half of them are on the one, and half on the other surface. Let a_1, a_2, a_3, a_4 be the four nodes on T. The joining line of any two of these points lies entirely on T, and is accordingly tangent to F in a third point. Hence, *the tetrahedron whose vertices are the nodes of the cubic surface T which touches F along s is inscribed to F as to its vertices and circumscribed to F as to its edges.* There are evidently ∞^3 such tetrahedra.

The four points a_i, and the points p_i counted twice, form the complete intersection of the cubic c with F.

The cubic surfaces S and S' defining the sextics s and s' intersect in six lines and a twisted cubic k. The latter passes through the four points p_i and the four vertices of the tetrahedron of reference. Conversely, every twisted cubic k passing through the vertices of the tetrahedron of reference intersects F in four remaining points p_i, which form the four points of intersection of two sextics s and s' (and hence are the common points of a singly infinite system of sextics.)

The Hessian F is invariant for the birational transformation T_5 defined by the equations

$$aw' : bx' : cy' : dz' = \frac{1}{w} : \frac{1}{x} : \frac{1}{y} : \frac{1}{z}.$$

This transformation interchanges the family of cubics k with the lines of space, and the sextics s with the plane quartics of F.* Four other transformations of a similar character may be obtained by starting from any of the other four tetrahedra contained in the fundamental pentahedron. These are

$$T_1: \quad bx' : cy' : dz' : ew' = \frac{1}{x} : \frac{1}{y} : \frac{1}{z} : \frac{1}{u},$$

$$T_2: \quad aw' : cy' : dz' : eu' = \frac{1}{w} : \frac{1}{y} : \frac{1}{z} : \frac{1}{u},$$

$$T_3: \quad aw' : bx' : dz' : eu' = \frac{1}{w} : \frac{1}{x} : \frac{1}{z} : \frac{1}{u},$$

$$T_4: \quad aw' : bx' : cy' : eu' = \frac{1}{w} : \frac{1}{x} : \frac{1}{y} : \frac{1}{u}.$$

These five operations, each of period 2, generate a group G of infinite order, since the operation T_iT_j is of infinite period. As far as the points of F are concerned, each transformation T_i has exactly the same effect. In other words, *there exists a subgroup of index* 2 *under the group* G *for which each point of* F *is unchanged in position.*

Since a plane quartic has 28 bitangents it follows that the sextic s has 28 bitangent cubics k out of each of the five families of such cubics. Suppose the plane of the quartic q corresponding to s to be rotated about one of the bitangents of the curve. The intersections of the moving plane with F will form a single infinity of plane quartics having the same line for bitangent. Hence, for a given sextic s there are 28 subfamilies of ∞^1 sextics, each family being tangent to s in a pair of points, or in other words, the four points p_i fall together in pairs 28 times on a given sextic.

Two sextics do not in general touch each other at a node of F. There are ∞^2 sextics which touch each other and have a given generator of the tangent cone at the node for a common tangent line.

Suppose that we consider the ∞^1 of these curves which are also mutually tangent at a second node of F. If the joining line of the first and second nodes is a line of F, then these curves are also mutually tangent in two other nodes of F, the four nodes together forming the vertices of one of the five fundamental tetrahedra. If the first and second nodes do not have a line of F for joining line then the two remaining points of intersection determine a line which meets two of the lines of F. For example, if the two nodes be

* Cf. Salmon, Geom. of Three Dim., 4th ed., p. 495.

wxu and *yzu* then the joining line of the two remaining points of intersection of the sextics so determined will meet F elsewhere in the two lines *wx* and *yz*.

CORNELL UNIVERSITY,
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NOTE ON THE GROUP OF ISOMORPHISMS.

BY DR. G. A. MILLER.

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LET $s_1, s_2, \cdots, s_g$ represent all the operators of a group G and let $t_\alpha s_\alpha$ correspond to s_α ($\alpha = 1, 2, \cdots, g$) in any given simple isomorphism of G with itself. It is evident that t_α is some operator of G. When G is abelian these t_α's must constitute a group T which is isomorphic with G.* In this isomorphism, t_α evidently can not be the inverse of s_α unless $s_\alpha = 1$. As this condition is sufficient as well as necessary, we have

THEOREM I.—*Every simple isomorphism of an abelian group A with itself may be obtained by 1° making A isomorphic with one of its subgroups or with itself in such a manner that no operator corresponds to its inverse, and 2° making each operator of A correspond to itself multiplied by the operator which corresponds to it in the given isomorphism.*

The simplest case that can present itself is the one in which the subgroup of G, which corresponds to identity of T in the given isomorphism between G and T, includes T. The resulting simple isomorphism of G with itself must correspond to an operator in the group of isomorphisms of G, whose order is equal to the operator of highest order in T. When the order of T is an odd prime number p, or the double of an odd prime, only one other case can present itself; viz, the case in which T corresponds to itself, or to its subgroup of an odd prime order, in the given isomorphism between G and T. The resulting simple isomorphism of G with itself may clearly correspond to a cyclical group of order $p - 1$, or to any one of its subgroups in the group of isomorphisms of G. These results lead to the following

* When G is non-abelian, these t_α's need not constitute a group, as can be seen from the simple isomorphisms of the symmetric group of order 6 with itself.