THE KNOWN FINITE SIMPLE GROUPS.

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THE list of systems of simple groups given in the BULLETIN for May, 1897, may be enlarged by the addition of a number of new systems determined by the writer during the past two years. By request a table has been constructed which should aid in the determination of the status of a newly discovered simple group. The first part of the table gives the orders of all simple groups of order less than one million which are included in the various systems except the first. In extending the table to the orders between one million and one billion, I have excluded, for the sake of brevity, the simple groups $LF(2,p^n)$ of type (3) having the orders

$$(p^{2n}-1)p^n \quad \text{for } p=2,$$

$$\tfrac{1}{2}(p^{2n}-1)p^n \quad \text{for } p>2.$$

But it is readily determined whether or not a given number N is of one of these two forms. Indeed, if q^s be the largest power of a prime contained in N, it must equal p^n or p^n+1. Hence, if q^s-1 be a power of a prime, then $p^n=q^s-1$; in the contrary case, $p^n=q^s$.

The calculations have all been made in duplicate; furthermore, care has been taken to include all orders coming within the assigned limits of the table.

In the following systems of simple groups, p denotes a prime number, m and n denote integers, each subject only to the limitations given.

(1) $$p,$$

(2) $$\tfrac{1}{2}n\,! \quad (n>4),$$

(3) $$LF(m,p^n)\equiv\frac{1}{d}(p^{nm}-1)p^{n(m-1)}(p^{n(m-1)}-1)p^{n(m-2)} \cdots (p^{2n}-1)p^n$$

where $p^n>3$ if $m=2$. Here d denotes the greatest common divisor of m and p^n-1.

$$(4)\quad HO(m, p^{2n}) \equiv \frac{1}{g}\,[p^{nm} - (-1)^m]\,p^{n(m-1)}\,[p^{n(m-1)} - (-1)^{m-1}]\,p^{n(m-2)} \cdots [p^{2n} - 1]p^n$$

where $p^{2n} > 3^2$ if $m = 2$ and $p^{2n} \neq 2^2$ if $m = 3$. Here g denotes the greatest common divisor of m and $p^n + 1$.

$$(5)\quad A(2m, p^n) \equiv \frac{1}{a}(p^{2nm} - 1)p^{n(2m-1)}(p^{n(2m-2)} - 1)p^{n(2m-3)} \cdots (p^{2n} - 1)p^n$$

where $p^n > 3$ if $m = 1$ and $p^n \neq 2$ if $m = 2$. Here $a = 1$ if $p = 2$ and $a = 2$ if $p > 2$.

$$(6)\quad FH(2m, 2^n) \equiv (2^{nm} - 1)\,[(2^{2n(m-1)} - 1)2^{2n(m-1)}]\,[(2^{2n(m-2)} - 1)2^{2n(m-2)}] \cdots [(2^{2n} - 1)2^{2n}]$$

where $m > 2$.

$$(7)\quad SH(2m, 2^n) \equiv (2^{nm} + 1)\,[(2^{2n(m-1)} - 1)2^{2n(m-1)}] \cdots [(2^{2n} - 1)2^{2n}]$$

where $m > 1$.

$$(8)\quad O(2m+1, p^n) \equiv \tfrac{1}{2}(p^{2nm} - 1)p^{n(2m-1)}(p^{n(2m-2)} - 1)p^{n(2m-3)} \cdots (p^{2n} - 1)p^n$$

where $p > 2$ and where $p^n \neq 3$ if $2m + 1 = 3$.

$$(9)\quad O(2m, p^n) \equiv \tfrac{1}{4}\,[p^{n(2m-1)} - \varepsilon^m p^{n(m-1)}]\,(p^{n(2m-2)} - 1)p^{n(2m-3)} \cdots (p^{2n} - 1)p^n$$

where $p > 2$ and $2m > 4$.

$$(10)\quad NS(2m, p^n) \equiv \tfrac{1}{2}\,[p^{n(2m-1)} + \varepsilon^m p^{n(m-1)}]\,(p^{n(2m-2)} - 1)p^{n(2m-3)} \cdots (p^{2n} - 1)p^n$$

where $p > 2$ and $2m > 2$.

In (9) and (10), $\varepsilon = \pm 1$ according as p^n is of the form $4l \pm 1$.

The group (3) is obtained* from the decomposition of the general linear homogeneous group on m indices in the $GF[p^n]$; concretely, it is the linear fractional group of determinant unity. The largest linear homogeneous group in m variables in the $GF[p^{2n}]$ leaving invariant

* Writer's Dissertation, *Annals of Mathematics*, 1897.

$$\xi_1^{p^n+1} + \cdots + \xi_m^{p^n+1}$$

may be called the *hyperorthogonal* group, since it has as a subgroup the general orthogonal group on m indices in the $GF[p^n]$. Its decomposition leads* to the simple group (4). The group (5) is obtained† from the abelian group on $2m$ indices in the $GF[p^n]$. Groups (6) and (7) are subgroups of index two under the generalized first‡ and second§ hypoabelian groups respectively; viz., linear groups on $2m$ indices in the $GF[2^n]$. Groups (8) and (9) are obtained from the decomposition of the orthogonal group in the $GF[p^n]$. Group (10) is obtained from the group defined by the invariant

$$\sum_{i=1}^{2m-1} \xi_i^2 + \nu\xi_{2m}^2$$

where ν is a not-square in the $GF[p^n]$. For the last three groups I refer to the memoir recently presented to the *American Journal of Mathematics.* Aside from certain low cases the results for the groups (8) and (9) had been previously‖ determined by the writer employing different methods.

Between certain groups given by the above systems there exist simple isomorphisms. The group $A(2, p^n)$ is identical with $LF(2, p^n)$. The groups $HO(2, p^{2n})$ and $O(3, p^n)$ are isomorphic with $LF(2, p^n)$. Indeed, the first is the so-called "imaginary form" of the third. We will, therefore, not mention $A(2, p^n)$ and $HO(2, p^{2n})$ in our tables. $SH(4, 2^n)$ is isomorphic with $LF(2, 2^{2n})$; $NS(4, p^n)$ with $LF(2, p^{2n})$ if $p \neq 2$. According as p^n is of the form 2^n, $4l+1$, or $4l-1$, the simple group designated by $LF(4, p^n)$ is simply isomorphic with the simple group designated by $FH(6, 2^n)$, $O(6, p^n)$, or $NS(6, p^n)$, respectively. I have not yet completed an investigation which promises to show that, according as $p^n = 2^n$, $4l+1$, $4l-1$, the group $HO(4, p^{2n})$ is simply isomorphic with the group $SH(6, 2^n)$, $NS(6, p^n)$, $O(6, p^n)$, respectively. I have succeeded in proving that the groups $HO(4, 2^2)$ and $SH(6, 2)$ are simply isomorphic, and have set up an abstract group and a substitution group on 36 letters simply isomorphic to each of the two linear groups. As a special case of the result referred

* The paper proving this result was offered January 22 to the *Math. Annalen.*

† *Quar. Jour. of Math.*, No. 114, 1897.

‡ *Quar. Jour. of Math.*, No. 117, 1898; BULLETIN, July, 1898.

§ "Determination of the structure of all linear homogeneous groups in a Galois field which are defined by a quadratic invariant," offered January 3d to the *Amer. Jour. of Math.*

‖ BULLETIN, February and May, 1898; *Proc. of the Calif. Acad. of Sciences*, third series, vol. 1, Nos. 4 and 5.

to below, the groups $O(5, 3)$ and $A(4, 3)$ are simply isomorphic. We have, therefore, four simple groups of order 25920, simply isomorphic in sets of two. [In a paper presented March 9, before the London Mathematical Society, I have shown that all four groups are simply isomorphic.]

Of the four known simple groups of order 20160, it was noted above that $LF(4, 2)$ and $FH(6, 2)$ are isomorphic.* Jordan has proven † that $LF(4, 2)$ is simply isomorphic to the alternating group on 8 letters. A very elegant group-theoretic proof has been recently given by Moore, "Concerning the general equations of the seventh and eighth degrees," *Mathematische Annalen*, vol. 51, pp. 417–444, in particular pp. 435–6. Finally, the alternating group on 8 letters is *not* isomorphic to the group $LF(3, 2^2)$, proven simple in the writer's Dissertation. This result (not yet published) has been reached by Miss Schottenfels, of Chicago, under the direction of Professor Moore.

The groups $O(5, p^n)$ and $A(4, p^n)$ are simply isomorphic.‡ Although the simple groups (8) and (5), for $p > 2$, have the same order, it seems probable that they are isomorphic only when $m = 1$ and 2; indeed, the corresponding continuous groups are not isomorphic if $m > 2$.

Aside from the cases just discussed, no coincidence in the orders of the simple groups in the above ten systems has been observed by the writer. Moreover, for the sets of values of m, n, p excluded in the above systems, there occur no simple groups of composite order in the series of factor groups for the corresponding linear groups, exceptions to this statement being the first hypoabelian group on $2m = 4$ indices and the orthogonal group on $2m = 4$ indices and the abelian group on 4 indices modulo 2. In the first of these three cases, the group has two of its factor groups identical with $LF(2, 2^n)$; in the second case, the group has two of its factor groups identical with $LF[2, p^n]$, $p > 2$; in the third case, the group is isomorphic to the alternating group on six letters.§

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* The general theorem, of which the result quoted is a special case, is given in the paper, "The structure of certain linear groups with quadratic invariants," communicated first to Professor Moore, May 11, 1898, and received for publication by the London Mathematical Society, September 1, 1898.

† Traité des Substitutions, No. 516.

‡ Dickson, "Concerning the abelian and hypoabelian groups," BULLETIN, p. 332, April, 1899; to be published in full in the Transactions of the Society.

§ Jordan, Traité des Substitutions, § 335. The writer has recently found a simple group theoretic proof, based on a correspondence of the generators and generational relations of the two groups.

In the following table of simple groups, a known simple isomorphism between two groups is indicated by the equality sign, a known non-isomorphism by the inequality sign.

Order of simple groups.		Character of simple groups.
60	$2^2.3.5$	$LF(2,2^2)=LF(2,5)=O(3,5)=SH(4,2)=\frac{1}{2}5!$
168	$2^3.3.7$	$LF(2,7)=O(3,7)=LF(3,2)$
360	$2^3.3^2.5$	$LF(2,3^2)=O(3,3^2)=NS(4,3)=\frac{1}{2}6\,!$
504	$2^3.3^2.7$	$LF(2,2^3)$
660	$2^2.3.5.11$	$LF(2,11)=O(3,11)$
1 092	$2\ .3.7.13$	$LF(2,13)=O(3,13)$
2 448	$2^4.3^2.17$	$LF(2,17)=O(3,17)$
2 520	$2^3.3^2.5.7$	$\frac{1}{2}7\,!$
3 420	$2^2.3^2.5.19$	$LF(2,19)=O(3,19)$
4 080	$2^4.3.5.17$	$LF(2,2^4)=SH(4,2^2)$
5 616	$2^4.3^3.13$	$LF(3,3)$
6 048	$2^5.3^3.7$	$HO(3,3^2)$
6 072	$2^3.3.11.23$	$LF(2,23)=O(3,23)$
7 800	$2^3.3.5^2.13$	$LF(2,5^2)=O(3,5^2)=NS(4,5)$
7 920	$2^4.3^2.5.11$	Cole, *Quarterly Journal*, vol. 27, p. 48.
9 828	$2^2.3^3.7.13$	$LF(2,3^3)=O(3,3^3)$
12 180	$2^2.3.5.7.29$	$LF(2,29)=O(3,29)$
14 880	$2^5.3.5.31$	$LF(2,31)=O(3,31)$
20 160	$2^6.3^2.5.7$	$\frac{1}{2}8\,!=LF(4,2)=FH(6,2)\neq LF(3,2^2)$
25 308	$2^2.3^2.19.37$	$LF(2,37)=O(3,37)$
25 920	$2^6.3^4.5$	$A(4,3)=O(5,3)=SH(6,2)=HO(4,2^2)$
32 736	$2^5.3.11\ 31$	$LF(2,2^5)$
34 440	$2^3.3.5.7.41$	$LF(2,41)=O(3,41)$
39 732	$2^2.3.7.11.43$	$LF(2,43)=O(3,43)$
51 888	$2^4.3.23.47$	$LF(2,47)=O(3,47)$
58 800	$2^4.3.5^2.7^2$	$LF(2,7^2)=O(3,7^2)=NS(4,7)$
62 400	$2^6.3.5^2.13$	$HO(3,2^4)$
74 412	$2^2.3^3.13.53$	$LF(2,53)=O(3,53)$
95 040	$2^6.3^3.5.11$	Substitution Group (Mathieu, Miller)
102 660	$2^2.3.5.29.59$	$LF(2,59)=O(3,59)$
113 460	$2^2.3.5.31.61$	$LF(2,61)=O(3,61)$
126 000	$2^4.3^2.5^3.7$	$HO(3,5^2)$
150 348	$2^2.3.11.17.67$	$LF(2,67)=O(3,67)$
178 920	$2^3.3^2.5.7.71$	$LF(2,71)=O(3,71)$
181 440	$2^6.3^4.5.7$	$\frac{1}{2}9\,!$
194 472	$2^3.3^2.37.73$	$LF(2,73)=O(3,73)$
246 480	$2^4.3.5.13.79$	$LF(2,79)=O(3,79)$
262 080	$2^6.3^2.5.7.13$	$LF(2,2^6)=SH(4,2^3)$
265 680	$2^4.3^4.5.41$	$LF(2,3^4)=O(3,3^4)=NS(4,3^2)$
285 852	$2^2.3.7.41.83$	$LF(2,83)=O(3,83)$
352 440	$2^3.3^2.5.11.89$	$LF(2,89)=O(3,89)$
372 000	$2^5.3.5^3.31$	$LF(3,5)$
456 288	$2^5.3.7^2.97$	$LF(2,97)=O(3,97)$
515 100	$2^2.3.5^2.17.101$	$LF(2,101)=O(3,101)$
546 312	$2^3.3.13.17.103$	$LF(2,103)=O(3,103)$
612 468	$2^2.3^3.53.107$	$LF(2,107)=O(3,107)$
647 460	$2^2.3^3.5.11.109$	$LF(2,109)=O(3,109)$
721 392	$2^4.3.7.19.113$	$LF(2,113)=O(3,113)$
885 720	$2^3.3.5.11^2.61$	$LF(2,11^2)=O(3,11^2)=NS(4,11)$

Order of simple groups.		Character of simple groups.
976 500	$2^2.3^2.5^3.7.31$	$LF(2,5^3) = O(3,5^3)$
979 200	$2^8.3^2.5^2.17$	$A(4,2^2)$
1 451 520	$2^9.3^4.5.7$	$A(6,2)$
1 814 400	$2^7.3^4.5^2.7$	$\frac{1}{2}10\,!$
1 876 896	$2^5.3^2.7^3.19$	$LF(3,7)$
3 265 920	$2^7.3^6.5.7$	$HO(4,3^2)$, $O(6,3)$
4 680 000	$2^6.3^2.5^4.13$	$A(4,5)$
5 515 776	$2^9.3^4.7.19$	$HO(3,2^6)$
5 663 616	$2^7.3.7^3.43$	$HO(3,7^2)$
6 065 280	$2^7.3^6.5.13$	$LF(4,3)$
9 999 360	$2^{10}.3^2.5.7.31$	$LF(5,2)$
13 685 760	$2^{10}.3^5.5.11$	$HO(5,2^2)$
16 482 816	$2^9.3^2.7^2.73$	$LF(3,2^3)$
19 958 400	$2^7.3^4.5^2.7.11$	$\frac{1}{2}11\,!$
42 456 960	$2^7.3^6.5\ 7.13$	$LF(3,3^2)$
42 573 600	$2^5.3^6.5^2.73$	$HO(3,3^4)$
70 915 680	$2^5.3^2.5.11^3.37$	$HO(3,11^2)$
138 297 600	$2^8.3^2.5^2.7^4$	$A(4,7)$
174 182 400	$2^{12}.3^5.5^2.7$	$FH(8,2)$
197 406 720	$2^{12}.3^4.5.7.17$	$SH(8,2)$
212 427 600	$2^4.3.5^2.7.11^3.19$	$LF(3,11)$
239 500 800	$2^9.3^5.5^2.7.11$	$\frac{1}{2}12\,!$
270 178 272	$2^5.3^2.7.13^3.61$	$LF(3,13)$
811 273 008	$2^4.3.7^2.13^3.157$	$HO(3,13^2)$
987 033 600	$2^{12}.3^4.5^2.7.17$	$LF(4,2^2) = FH(6,2^2)$

I added a few simple groups of order greater than a billion, but following immediately those given in the table.

$LF(4,5) = 2^7.3^2.5^6.13.31,$
$LF(5,3) = 2^9.3^{10}.5.11^2.13,$
$LF(5,5) = 2^{11}.3^3.5^{10}.7.13.31.61,$
$HO(4,2^4) = 2^{12}.3^2.5^3.13.17,$
$HO(5,3^2) = 2^{11}.3^{10}.5.7.61,$
$A(4,2^3) = 2^{12}.3^4.5.7^2.13,$
$A(4,11) = 2^6.3^2.5^2.11^4.61,$
$A(6,2^2) = 2^{18}.3^4.5^3.7.13.17,$
$A(8,2) = 2^{16}.3^5.5^2.7.17,$
$FH(6,2^3) = 2^{18}.3^4.5.7^3.13.73,$
$FH(10,2) = 2^{20}.3^5.5^2.7.17.31,$
$SH(6,2^2) = 2^{12}.3^2.5^3.13.17,$
$NS(6,5) = 2^7.3^4.5^6.7.13,$
$NS(8,3) = 2^{10}.3^{12}.5.7.13.41,$
$O(6,7) = 2^{10}.3^2.5^2.7^6.43,$
$O(8,3) = 2^{12}.3^{12}.5^2.7.13,$

$LF(4,3^2) = 2^{10}.3^{12}.5^2.7.13.41,$
$LF(5,2^2) = 2^{20}.3^5.5^2.7.11.17.31,$
$LF(6,2) = 2^{15}.3^4.5.7^2.13,$
$HO(4,5^2) = 2^7.3^4.5^6.7.13,$
$HO(6,2^2) = 2^{15}.3^7.5.7.11,$
$A(4,3^2) = 2^8.3^8.5^2.41,$
$A(6,3) = 2^9.3^9.5.7.13,$
$A(6,5) = 2^9.3^4.5^9.7.13.31,$
$A(8,3) = 2^{14}.3^{16}.5^2.7.13.41,$
$FH(8,2^2) = 2^{24}.3^5.5^4.7.13.17^2,$
$SH(10,2) = 2^{20}.3^6.5^2.7.11.17,$
$NS(6,3^2) = 2^9.3^{12}.5^3.41.73,$
$NS(8,5) = 2^{10}.3^4.5^{12}.7.13.31.313,$
$O(6,11) = 2^7.3^4.5^2.11^6.37.61,$
$O(8,5) = 2^{12}.3^5.5^{12}.7.13^2.31.$

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