THE LARGEST LINEAR HOMOGENEOUS GROUP WITH AN INVARIANT PFAFFIAN.

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1. IN the December number of the BULLETIN (pp. 120–135) I have shown that the second compound of the general $2m$-ary linear homogeneous group is a linear group in $C_{2m,2} \equiv m(2m-1)$ variables which leaves invariant the Pfaffian

$$F \equiv [1, 2, \cdots, 2m].$$

Denoting the variables as follows :

$$(1) \qquad Y_{ij} \equiv - Y_{ji} \qquad (i, j = 1, \cdots, 2m\,;\ i \neq j),$$

the second compound was proved to contain exactly $(2m)^2$ linearly independent infinitesimal transformations

$$(2) \qquad \sum_{r \neq s,\, t}^{r=1, \cdots, 2m} Y_{rt} \frac{\partial f}{\partial Y_{rs}} \delta t. \qquad (t, s = 1, \cdots, 2m).$$

The object of the present note is to prove that the largest linear homogeneous group G in the $m(2m-1)$ variables (1) which leaves invariant the Pfaffian F contains only the $(2m)^2$ linearly independent transformations (2).

2. Let the general infinitesimal transformation of the group G be as follows:

$$(3) \qquad \delta Y_{ij} = \sum_{k \neq l}^{k=1, \cdots, 2m} a^{ij}_{kl} Y_{kl} \delta t \qquad (i, j = 1, \cdots, 2m\,;\ i \neq j),$$

where, on account of (1), we may suppose

$$(4) \qquad a^{ij}_{kl} = - a^{ij}_{lk} = + a^{ji}_{lk}.$$

The condition that (3) shall multiply F by a constant $c\delta t$ is as follows :

$$(5) \qquad \sum_{i,j,k,l} \frac{\partial F}{\partial Y_{ij}} a^{ij}_{kl} Y_{kl} = cF.$$

Now

$$\frac{\partial F}{\partial Y_{ij}} = \frac{\partial}{\partial Y_{ij}} \{(-1)^{i+j-1}[ij\,1\,2 \cdots i-1\,i+1 \cdots j-1\,j+1 \cdots 2m]$$

$$= (-1)^{i+j-1}\,[1\,2 \cdots i-1\;i+1 \cdots j-1\,j+1 \cdots 2m].$$

Comparing the coefficients of the terms in (5) of the type

$$(-1)^{\sigma}\, Y_{i_1 i_2} Y_{i_3 i_4} \cdots Y_{i_{2m-1} i_{2m}},$$

where $i_1, i_2, \cdots, i_{2m}$ is a permutation of $1, 2, \cdots, 2m$ and where σ denotes the number of transpositions giving that permutation, we obtain the conditions

(6) $$a^{i_1 i_2}_{i_1 i_2} + a^{i_3 i_4}_{i_3 i_4} + \cdots + a^{i_{2m-1} i_{2m}}_{i_{2m-1} i_{2m}} = c.$$

Comparing the coefficients of the terms,

$$Y^2_{i_3 i_4} Y_{i_5 i_6} \cdots Y_{i_{2m-1}\, i_{2m}}$$

we obtain the conditions

(7) $$a^{i_1 i_2}_{i_3 i_4} = 0 \qquad (i_3 \text{ and } i_4 \neq i_1 \text{ or } i_2).$$

Comparing the coefficients of the terms

$$Y_{i_1 i_3} Y_{i_3 i_4} \cdots Y_{i_{2m-1}\, i_{2m}},$$

we find

(8) $$a^{i_1 i_2}_{i_1 i_3} - a^{i_4 i_2}_{i_4 i_3} = 0$$

(i_1, i_2, i_3, i_4, being any four different integers $\leqq 2m$).

We may now obtain a complete set of linearly independent infinitesimal transformations (3), which leave F invariant. According as every $a^{i_1 i_2}_{i_3 i_4}$ is zero, or not every such a is zero, we obtain two independent types of transformations (3), which together form the desired complete set. We consider the two types in succession:

(a) If any $a^{rs}_{rt} \neq 0$, say $= 1$, where r, s, t are distinct integers $\leqq 2m$, then by (8) we have

$$a^{rs}_{rt} = 1 \quad (r = 1, \cdots, 2m\,;\; r \neq s, t).$$

Setting every other $a = 0$, we obtain a set of solutions of (6), (7), (8), for which

$$\begin{cases} \delta Y_{rs} = Y_{rt}\delta t & (r = 1, \cdots, 2m;\; r \neq s, t) \\ \delta Y_{ij} = 0 & (i, j = 1, \cdots, 2m;\; i \neq r). \end{cases}$$

We thus obtain the $2m\ (2m-1)$ infinitesimal transformations (included in the formula (2))

$$(9)\qquad \sum_{r \neq s,\, t}^{r=1, \cdots, 2m} Y_{rt} \frac{\partial f}{\partial Y_{rs}} \delta t \qquad \left(\begin{matrix} s, t = 1, \cdots, 2m \\ s \neq t \end{matrix} \right),$$

which are therefore linearly independent.

(b) If next

$$a_{rt}^{rs} = 0 \quad (r,\ s,\ t = 1, \cdots, 2m\ ;\ r \neq s, \neq t),$$

the general transformation (3) becomes

$$\delta Y_{ij} = a_{ij}^{ij} Y_{ij} \delta t \qquad (i, j = 1, \cdots, 2m),$$

where the a_{ij}^{ij} are subject to the conditions (6).

Writing for brevity [see (4)],

$$a_{ij}^{ij} \equiv (ij) = (ji),$$

these conditions (6) become

$$(6)\qquad (i_1 i_2) + (i_3 i_4) + \cdots + (i_{2m-1} i_{2m}) = c.$$

We obtain at once the following $2m$ sets of solutions of these equations, each set being given by one value of l chosen from $1, 2, \cdots, 2m$;

$$(10) \left\{ \begin{array}{l} (l1) = (l2) = \cdots = (l\,l-1) = (l\,l+1) = \cdots = (l\,2m) = c \\ (ij) = 0 \qquad\qquad\qquad [i, j = 1, \cdots, 2m\ ;\ i \neq l]. \end{array} \right.$$

These sets of solutions of the equations (6), (7), (8) give rise to the following $2m$ infinitesimal transformations :

$$(11)\qquad A_{ll} \equiv \sum_{j \neq l}^{j=1, \cdots, 2m} Y_{lj} \frac{\partial f}{\partial Y_{lj}} \qquad (l = 1, \cdots, 2m).$$

These transformations are linearly independent if $m \gtreqless 2$. Indeed, if

$$\sum_{l=1}^{2m} k_l A_{ll} = 0,$$

upon equating the coefficients of $\dfrac{\partial f}{\partial Y_{rs}}$ in the two members, we have

$$k_r + k_s = 0 \qquad (r, s = 1, \cdots, 2m\ ;\ r \neq s).$$

Hence, if $m \gtreqless 2$, $k_l = 0\ (l = 1, \cdots, 2m)$.

The transformations (9) and (11) make up the $(2m)^2$ linearly independent transformations (2). It follows from the theorem of the next paragraph that there do not exist more than $2m$ linearly independent transformations of the type (b). We will then have proved the following theorem :

The largest linear homogeneous group in $C_{2m,2}$ variables leaving invariant the Pfaffian $[1, 2, \cdots, 2m]$ *is identical with the second compound of the general m-ary linear homogeneous group.*

3. THEOREM. *The* $m(2m-1)$ *quantities*

$$(ij) \equiv (ji) \qquad [i, j = 1, 2, \cdots, 2m, i \neq j]$$

satisfying the $1 \cdot 3 \cdot 5 \cdots (2m-3)(2m-1)$ *equations*

$$[E_{2m}] \qquad (i_1 i_2) + (i_3 i_4) + \cdots + (i_{2m-1} i_{2m}) = c_{2m}$$

can all be expressed in terms of certain $2m$ *of the* (ij), *for example,*

$$[Q_{2m}] \begin{cases} (1\ 2), (3\ 4), (5\ 6), \cdots, (2l-1\ 2l), \cdots, (2m-1\ 2m); \\ (2\ 3); (2\ 4), (4\ 6), \cdots, (2l-2\ 2l), \cdots, (2m-2\ 2m), \end{cases}$$

but not in terms of fewer than $2m$ *of them if* $m > 1$.

The last part of the theorem follows from the linear independence of the $2m$ infinitesimal transformation of type (b) above.

The first part of the theorem will be proved by induction. For $m = 2$, it is evident; for the equations $[E_4]$ are as follows:

$$(12) + (34) = (13) + (24) = (14) + (23) = c_4.$$

Supposing the first part of the theorem to be true for a given value of $2m$, we can prove it true for the next value $2(m+1)$. Indeed, applying this hypothesis to certain equations of the set $[E_{2m+2}]$, viz.:

$$(i_1 i_2) + (i_3 i_4) + \cdots + (i_{2m-1}\, i_{2m}) = c_{2m+2} - (2m+1\ \ 2m+2),$$

where $i_1, i_2, \cdots, i_{2m}$ is a permutation of $1, 2, \cdots, 2m$, it follows that the quantities

$$(ij) \qquad [i, j = 1, 2, \cdots, 2m\,;\ i \neq j]$$

can be expressed in terms of the quantities Q_{2m} and that c_{2m+2} is expressible in terms of the quantities Q_{2m} together with $(2m+1\ 2m+2)$.

Consider next the equations of the set $[E_{2m+2}]$

$$(i_1 i_2) + (i_3 i_4) + \cdots + (i_{2m-3}\, i_{2m-2}) + (j\ 2m+1)$$
$$+ (2m\ 2m+2) = c_{2m+2},$$

where $i_1, i_2, \cdots, i_{2m-2}, j$ form a permutation of $1, 2, \cdots, 2m-1$. It follows that every

$$(j\ 2m+1) \qquad [j = 1, 2, \cdots, 2m-1]$$

is expressible in terms of the Q_{2m}, c_{2m+2} and $(2m\ 2m+2)$ and hence in terms of the Q_{2m+2}.
From the equation

$$(1\ 2) + (3\ 4) + \cdots + (2m-5\ 2m-4) + (2m-3\ 2m)$$
$$+ (2m-2\ 2m+1) + (2m-1\ 2m+2) = c_{2m+2}$$

we have $(2m-1\ 2m+2)$ expressed in terms of the quantities Q_{2m+2} (by using our earlier results). Hence, from the equation

$$(1\ 2) + (3\ 4) + \cdots + (2m-3\ 2m-2) + (2m\ 2m+1)$$
$$+ (2m-1\ 2m+2) = c_{2m+2},$$

we obtain $(2m\ 2m+1)$ expressed in terms of the Q_{2m+2}. We have, therefore, every

$$(j\ 2m+1) \qquad [j = 1, 2, \cdots, 2m+2]$$

expressed in terms of the Q_{2m+2}.

Finally, from the equations

$$(i_1 i_2) + \cdots + (i_{2m-1} i_{2m}) + (j\ 2m+2)$$
$$+ (2m-1\ 2m+1) = c_{2m+2},$$

where $j, i_1, i_2, \cdots, i_{2m}$ form a permutation of $1, 2, \cdots, 2m-2, 2m$, we are able to express

$$(j\ 2m+2) \qquad [j = 1, 2, \cdots, 2m-2, 2m]$$

in terms of the Q_{2m+2}.

Combining our results, we find that every

$$(i\,j) \qquad [i, j = 1, 2, \cdots, 2m+2; i \neq j]$$

is expressible in terms of the quantities Q_{2m+2}.

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