DETERMINANTS OF QUATERNIONS.

BY PROFESSOR JAMES MILLS PEIRCE.

(Abstract, read before the American Mathematical Society at the Meeting of February 25, 1899.)

I ADOPT as the basis of the following discussion the convention that the order of factors in every term of a determinant is the same as the order of columns in the matrix; while the order of ranks in each term fixes the sign of the term, according to the usual rule for determinants of scalars. It follows that, in quaternions, columns and ranks are not interchangeable; that the transposition of ranks causes the same changes of sign as in scalar algebra; and that a determinant which has two equal ranks vanishes; but that the position of a column is, in general, fixed; and that a determinant may have two or more equal columns without vanishing. If, however, a determinant has a scalar column, such column may be displaced with the same changes of sign as in scalar algebra; and if two scalar columns of a matrix are equal, the determinant vanishes.

The usual development of a determinant as a sum of products of constituents from any row into corresponding minors can be employed only for the first and last columns of a determinant of quaternions, or for a scalar column or rank; but a more general formula, preserving the order of factors in each term, may be employed for *any* column or rank. The addition theorem holds for determinants of quaternions; but the multiplication theorem does not hold, since its proof involves the commutative property of multiplication. The latter theorem may be used, however, when one of the factors is a determinant of scalars; a case which may often occur in applications.

A *Repeating Determinant* is defined as a determinant of which every column is either equal or conjugate to an assumed standard column; and a *Resultant* of several quaternions, as a repeating determinant found from those quaternions, and divided by the number of its terms, that is by the factorial of its order. A single quaternion has two resultants, which are equal respectively to the quaternion and to its conjugate; and n quaternions have 2^n resultants, among which we distinguish two *uniform* resultants, of which the columns are all equal, and two *alternating* resultants, of which the columns are alternately normal and conjugate.

Since all scalars are self-conjugate, the two resultants of a single scalar are equal; so that one scalar has but one resultant, which is the scalar itself. Since a determinant having two equal scalar columns vanishes, *every resultant of two or more scalars is identically equal to zero.*

Since the conjugate of a vector is the same as its negative, the different resultants of a given system of vectors are all equal except in sign. Hence, in the case of vectors, we may confine ourselves to *uniform* resultants. We may now compute resultants of the successive orders, either by experiment, or (more instructively) by the aid of general rules of the calculus of quaternions. If the resultant of a system of quantities be denoted by writing the letter R. before the principal diagonal term of the determinant, we have, for the resultants of vectors,

$$\mathrm{R}.\alpha_1\alpha_2 = \tfrac{1}{2}(\alpha_1\alpha_2 - \alpha_2\alpha_1) = \mathrm{V}\alpha_1\alpha_2,$$

$$\mathrm{R}.\alpha_1\alpha_2\alpha_3 = \tfrac{1}{6}(\alpha_1\alpha_2\alpha_3 - \alpha_1\alpha_3\alpha_2 + \alpha_2\alpha_3\alpha_1 - \alpha_2\alpha_1\alpha_3$$
$$+ \alpha_3\alpha_1\alpha_2 - \alpha_3\alpha_2\alpha_1) = \mathrm{S}\,\alpha_1\alpha_2\alpha_3,$$

$$\mathrm{R}.\alpha_1\alpha_2\alpha_3\alpha_4 = \tfrac{1}{24}(\alpha_1\alpha_2\alpha_3\alpha_4 - \alpha_1\alpha_2\alpha_4\alpha_3 + \text{etc.}) = 0\,;$$

whence it follows, by the development in minors, that *every resultant of more than three vectors vanishes identically.*

The relation of the forms $\mathrm{V}\alpha_1\alpha_2$ and $\mathrm{S}\alpha_1\alpha_2\alpha_3$ to certain scalar determinant forms is well known. But it seems not to have been observed that they are *themselves determinants.*

To find the resultants of a system of n quaternions, we may break up each determinant into its scalar and vector parts, and then, by the addition theorem, resolve the required resultant into the sum of 2^n determinant parts. But, since every determinant vanishes, which has two equal scalar columns, there will remain only n determinants having one scalar column each, which are all equal save in sign, and *one* determinant of vectors. Thus, the computation is reduced to that of resultants of vectors of the orders n and $n - 1$.

We may confine ourselves here to *alternating* resultants, which are found to be the best adapted to use in theorems relating to linear equations. We shall find it convenient to denote the conjugate of a quaternion by a *dash* over the letter which denotes the quaternion; writing, for example,

$$\bar{p} = \mathrm{K}\,p.$$

Using the method of computation indicated above and denoting by a and α respectively the scalar and vector parts of a quaternion p, we now easily obtain the following results :

$$\mathrm{R}.p_1\bar{p}_2 = a_2\alpha_1 - a_1\alpha_2 - \mathrm{V}\alpha_1\alpha_2,$$

$$\mathrm{R}.p_1\bar{p}_2p_3 = -a_1\mathrm{V}\alpha_2\alpha_3 - a_2\mathrm{V}\alpha_3\alpha_1 - a_3\mathrm{V}\alpha_1\alpha_2 - \mathrm{S}\alpha_1\alpha_2\alpha_3,$$

$$\mathrm{R}.\bar{p}_1p_2\bar{p}_3p_4 = a_1\mathrm{S}\alpha_2\alpha_3\alpha_4 - a_2\mathrm{S}\alpha_1\alpha_3\alpha_4 + a_3\mathrm{S}\alpha_1\alpha_2\alpha_4 - a_4\mathrm{S}\alpha_1\alpha_2\alpha_3,$$

$$\mathrm{R}.p_1\bar{p}_2p_3\bar{p}_4p_5 = 0\ ;$$

whence it may be proved that *every resultant of more than four quaternions vanishes identically.* It will be found that a *uniform* resultant vanishes even for four quaternions.

It may now be shown that the vanishing of an alternating resultant of a system of quaternions is the necessary and sufficient condition of the existence of a linear equation with scalar coefficients between those quaternions ; but the vanishing of a *uniform* resultant is not in all cases a *sufficient*, though always a *necessary*, condition of the existence of the equation. In the case of a system of *vectors*, however, the vanishing of *any* resultant involves the vanishing of all the resultants. It is well known that $\mathrm{V}\alpha\beta = 0$ and $\mathrm{S}\alpha\beta\gamma = 0$ are respectively the conditions of the existence of equations of the forms

$$x\alpha + y\beta = 0, \quad \text{and} \quad x\alpha + y\beta + z\gamma = 0\ ;$$

and that *any four* vectors satisfy a linear equation. The conditions for quaternions are given in another (somewhat empirical) form by Hamilton. But their relation to the theory of determinants is not indicated and this seems to the writer to give the true key to the subject of linear equations.

The vanishing of a resultant of several quaternions is not only the *condition* of the existence of a linear equation, but it is *itself equivalent to that equation.* Thus the condition $\mathrm{R}.\alpha_1\alpha_2\alpha_3 = 0$, may be written

$$\tfrac{1}{3}(\alpha_1\mathrm{R}.\alpha_2\alpha_3 - \alpha_2\mathrm{R}.\alpha_1\alpha_3 + \alpha_3\mathrm{R}.\alpha_1\alpha_2) = 0,$$

or (removing the factor $\frac{1}{3}$),

$$\alpha_1\mathrm{V}\alpha_2\alpha_3 + \alpha_2\mathrm{V}\alpha_3\alpha_1 + \alpha_3\mathrm{V}\alpha_1\alpha_2 = 0\ ;$$

where, when $\mathrm{R}.\alpha_1\alpha_2\alpha_3 = 0$, the vector coefficients of α_1, α_2, α_3, have scalar ratios to each other, so that by removing a common vector factor, the desired linear equation with scalar coefficients is at once obtained. So (or more directly) in other cases.

ST. PETERSBURG, *January*, 1899.