THE THEOREMS OF OSCILLATION OF STURM AND KLEIN. (THIRD PAPER.)

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(Read before the American Mathematical Society at its Fifth Summer Meeting, Boston, Mass., August 19, 1898.)

THE following pages form a continuation of two papers presented under the same title to the Society and printed on pp. 295–313 and 365–376 of the preceding volume of the BULLETIN. These papers will be referred to as *Th. of Osc. 1* and *2* respectively.

The object of the present paper is to extend the results so far established to some cases in which the coefficients of the differential equation in question are no longer continuous throughout the intervals with which we are concerned. Such extensions are made in §§ 2 and 3 of the present paper but for this purpose it is necessary in § 1 to establish some fundamental theorems concerning linear differential equations of the second order with discontinuous coefficients, results which are perhaps of some interest apart from the special applications here made of them. Before entering on these questions, however, it will be convenient to describe accurately the kind of discontinuities with which we shall deal, and to establish certain general theorems.

All the functions with which we shall have to deal in the present paper are, throughout the interval in which we consider them, single valued *real* functions of one or more *real* variables. Taking first the case of a function of a single variable $f(x)$, we shall consider only the case in which this function has in an interval $a \leqq x \leqq b$ a finite number of points of discontinuity.

The simplest discontinuities from some points of view are the so-called *finite discontinuities** of $f(x)$, *i. e.*, discontinuities $x = c$ for which a positive quantity M can be found such that in the neighborhood of c, $|f(x)| < M$. Going beyond these finite discontinuities we have discontinuities at

* Strictly speaking this includes the case in which the discontinuity is simply due to the fact that the function has not been defined at the point in question; for example e^{-1/x^2} at the origin. We will, however, here once for all make the convention that in such cases we will regard the function as being so defined at the point in question as to preserve the continuity if possible.

which the function becomes infinite for some or for all methods of approach. For our purpose we need merely to restrict these discontinuities by requiring that the functions do not become infinite more strongly than $(x-c)^{-i}$ where $0<i<1$. We may then describe the discontinuities with which we have to deal, including the finite discontinuities, as follows:

(A) *The functions $f(x)$ with which we shall have to deal are such that a function $\psi(x)$ exists single valued and continuous throughout the interval $a \leqq x \leqq b$ and such that at every point of this interval except $c_1, c_2, \cdots, c_n$*

$$\psi(x) = [(x-c_1)(x-c_2)\cdots(x-c_n)]^i f(x),$$

where $0<i<1$.

It should be noticed that for functions of this sort $\int f(x)dx$ extended over a part or the whole of the interval ab is absolutely convergent, *i. e.*, $\int |f(x)|\,dx$ is convergent.

We shall also have to deal with functions of two or more independent variables $x, \lambda, \mu, \cdots$ which we can describe as follows:

(B) *The functions $f(x;\lambda,\mu,\cdots)$ with which we shall have to deal are such that a function ψ can be found which is a single valued and continuous function of $(x,\lambda,\mu,\cdots)$ when $a \leqq x \leqq b$, $\lambda_2 \leqq \lambda \leqq \lambda_1$, $\mu_2 \leqq \mu \leqq \mu_1, \cdots$ and such that for all of these values except when $x = c_1, c_2, \cdots, c_n$*

$$\psi(x;\lambda,\mu,\cdots) = [(x-c_1)(x-c_2)\cdots(x-c_n)]^i f(x;\lambda,\mu,\cdots)$$

where i is a constant satisfying the inequality $0<i<1$.

We give now an important theorem concerning such functions:

(C) *If $f(x;\lambda,\mu,\cdots)$ is a function of the kind described in* (B) *and if x_0 is a constant satisfying the inequality $a \leqq x_0 \leqq b$, then*

$$F(x;\lambda,\mu,\cdots) = \int_{x_0}^{x} f(x;\lambda,\mu,\cdots)dx$$

is a continuous function of $(x,\lambda,\mu,\cdots)$ for all the values in question.

In order to prove this we have merely to show that as $\Delta x, \Delta\lambda, \Delta\mu, \cdots$ approach zero independently of each other the difference:

$$\Delta F = F(x+\Delta x;\lambda+\Delta\lambda,\mu+\Delta\mu,\cdots) - F(x;\lambda,\mu,\cdots)$$

approaches zero. Now we can write

$$\Delta F = \int_{x_0}^{x+\Delta x} f(x;\ \lambda + \Delta\lambda,\ \mu + \Delta\mu, \cdots)\,dx - \int_{x_0}^{x+\Delta x} f(x;\ \lambda,\ \mu, \cdots)\,dx$$
$$+ \int_{x_0}^{x+\Delta x} f(x;\ \lambda,\ \mu, \cdots)\,dx - \int_{x_0}^{x} f(x;\ \lambda,\ \mu, \cdots)\,dx.$$

We have, therefore, the inequality

$$\Delta F \mid \leqq \int_{x_0}^{x+\Delta x} \mid f(x;\ \lambda + \Delta\lambda,\ \mu + \Delta\mu, \cdots) - f(x;\ \lambda,\ \mu, \cdots) \mid \cdot \mid dx \mid$$
$$+ \int_{x}^{x+\Delta x} \mid f(x;\ \lambda,\ \mu, \cdots) \mid \cdot \mid dx \mid.$$

The second of these integrals clearly approaches zero as Δx approaches zero. The first may be written, if we use the same notation as is used in (B), in the form

$$\int_{x_0}^{x+\Delta x} \mid (x - c_1) \cdots (x - c_n) \mid^{-i} \mid \Delta\psi \mid \mid dx \mid$$
$$\leqq \int_a^b \mid (x - c_1) \cdots (x - c_n) \mid^{-i} \mid \Delta\psi \mid dx,$$

where $\Delta\psi = \psi(x;\ \lambda + \Delta\lambda,\ \mu + \Delta\mu, \cdots) - \psi(x;\ \lambda,\ \mu, \cdots)$.

If now we denote the greatest value of $\mid \Delta\psi \mid$ in the interval $a \leqq x \leqq b$ by ε we see that the integral we are considering does not exceed

$$\varepsilon \int_a^b \mid (x - c_1) \cdots (x - c_n) \mid^{-i} dx.$$

But owing to the uniform continuity of ψ, ε clearly approaches zero when $\Delta\lambda$, $\Delta\mu$, $\cdots$ all approach zero. Accordingly the whole integral, and therefore ΔF, approaches zero, as was to be proved.

(D) *If $f(x;\ \lambda,\ \mu, \cdots)$ is a function of the kind described in* (B) *and $\varphi(x,\ \lambda,\ \mu, \cdots)$ is throughout the region in question a single valued continuous function of $(x, \lambda, \mu, \cdots)$, then $f.\varphi$ is a function of the kind described in* (B).

The proof of this theorem follows immediately from (B).

(E) *If throughout the interval $a \leqq x \leqq b$, $\varphi(x)$ is single valued and continuous and $\varphi(c) = 0$ $(a \leqq c \leqq b)$, then if $k > 0$ and x_0 is any point of ab*

$$\lim_{x=c} \left[(x - c)^k \int_{x_0}^{x} \frac{\varphi(x)}{(x - c)^{k+1}}\,dx \right] = 0.$$

In order to prove this let us take a point x_1 so near to c that when $|x-c| \leqq |x_1 - c|$, $|\varphi(x)| < k\varepsilon$. Now denoting the expression which stands above in square parentheses by $F(x)$ we have

$$F(x) = (x-c)^k \int_{x_0}^{x_1} \frac{\varphi(x)}{(x-c)^{k+1}} dx + (x-c)^k \int_{x_1}^{x} \frac{\varphi(x)}{(x-c)^{k+1}} dx$$

so that when $|x-c| < |x_1 - c|$

$$|F(x)| < |x-c|^k \int_{x_0}^{x_1} \frac{|\varphi(x)|}{|x-c|^{k+1}} |dx|$$

$$+ k\varepsilon |x-c|^k \int_{x_1}^{x} \frac{|dx|}{|x-c|^{k+1}}$$

$$= |x-c|^k \int_{x_0}^{x} \frac{|\varphi(x)|}{|x-c|^{k+1}} |dx| + \varepsilon \left[1 - \left(\frac{x-c}{x_1-c}\right)^k\right].$$

The second of these two terms is evidently less than ε while the first can be made less than ε by taking $|x-c|$ sufficiently small. No matter how small ε may be chosen $|F(x)|$ can be made less than 2ε by taking $|x-c|$ small enough, and this proves the theorem.

§ 1. *On Linear Differential Equations of the Second Order with Discontinuous Coefficients.*

Let us consider the differential equation

$$(1) \qquad \frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)\,y = 0,$$

in which throughout the interval $a \leqq x \leqq b$ the coefficients p and q are functions of the kind described in (A). We can, of course, not expect to find solutions y which satisfy (1) at *all* points of ab. We shall therefore seek functions which, together with their first derivatives are single valued and continuous throughout ab and except at the points of discontinuity ($x = c_1, c_2, \cdots, c_n$) of p or q satisfy (1).

In order to simplify matters let us first assume that there is only *one* point ($x = c$) of discontinuity in ab. It is at once clear from (A) that we can find a positive quantity M such that at all points of the interval $a \leqq x \leqq b$ (except c)

$$|p| < M|x-c|^{-i}, \quad |q| < M|x-c|^{-i}.$$

We will now prove that *the method of successive approximations enables us to find a function y which, together with its first derivative, is single valued and continuous throughout the interval $a \leqq x \leqq b$, which satisfies* (1) *at every point of this interval except at c, and which at c has the arbitrarily prescribed value γ while its derivative has the arbitrarily prescribed value γ'.**

To prove this we will let

$$Y_0 = \gamma + \gamma'(x - c)$$

and compute the quantities $Y_1, Y_2, \cdots$ from the formulæ

$$(2) \qquad \begin{aligned} Y_n' &= -\int_c^x [pY_{n-1}' + qY_{n-1}]\,dx, \\ Y_n &= \int_c^x Y_n'\,dx, \end{aligned}$$

where accents denote differentiation. It should be noticed that the functions Y_n' and Y_n as thus defined are continuous throughout the interval ab. We wish to prove that the series

$$(3) \qquad Y_0 + Y_1 + Y_2 + \cdots$$

converges throughout ab and represents the desired solution of (1). In order to prove this let us consider by the side of the series (3) the series formed by differentiating it term by term

$$(4) \qquad Y_0' + Y_1' + Y_2' + \cdots.$$

We will begin by proving (3) and (4) uniformly convergent throughout ab. Let us write for brevity $|x - c| = t$. Let l be a positive quantity greater than 1 and such that at all points of ab, $t < l$; and finally let C be a positive quantity such that at all points of ab the inequalities $|Y_0| < C$, $|Y_0'| < C$ hold. We will first establish the inequalities

$$(5) \qquad \begin{aligned} |Y_n'| &< \frac{C}{n!}\left(\frac{2Ml}{1-i}t^{1-i}\right)^n, \\ |Y_n| &< \frac{C}{n!}\left(\frac{2Ml}{1-i}t^{1-i}\right)^n. \end{aligned}$$

These formulæ evidently hold when $n = 0$. They will therefore be established for all values of n if assuming them

* A precisely similar theorem holds for homogeneous linear differential equations of order higher than the second. The proof is essentially the same as that here given.

to hold for a given value of n we can show that they hold for the next larger value. Now we have

$$|Y'_{n+1}| \leqq M \int_0^t t^{-i} [|Y_n'| + Y_n|] \, dt$$

$$< \frac{2CM}{n!} \left(\frac{2Ml}{1-i}\right)^n \int_0^t t^{n-(n+1)i} \, dt$$

$$= \frac{C}{(n+1)!} \left(\frac{2Ml}{1-i}\right)^{n+1} \frac{1}{l} t^{(n+1)(1-i)}$$

$$< \frac{C}{(n+1)!} \left(\frac{2Ml}{1-i} t^{1-i}\right)^{n+1}.$$

Furthermore

$$|Y_{n+1}| \leqq \int_0^t |Y'_{n+1}| \, dt < \frac{C}{(n+1)!} \left(\frac{2Ml}{1-i}\right)^{n+1} \frac{1}{l} \int_0^t t^{(n+1)(i-1)} dt$$

$$= \frac{C}{(n+1)! \, [(n+1)(1-i)+1]} \left(\frac{2Ml}{1-i}\right)^{n+1} \frac{t}{l} t^{(n+1)(1-i)}$$

$$< \frac{C}{(n+1)!} \left(\frac{2Ml}{1-i} t^{1-i}\right)^{n+1}.$$

Thus formulæ (5) are established. From these formulæ we can deduce the following inequalities in which the variable t no longer enters

$$(6) \qquad \begin{aligned} |Y_n'| &< \frac{C}{n!} \left(\frac{2Ml^{2-i}}{1-i}\right)^n, \\ |Y_n| &< \frac{C}{n!} \left(\frac{2Ml^{2-i}}{1-i}\right)^n. \end{aligned}$$

From (6) it appears that each term in (3) and also each term in (4) is less than the corresponding term in the exponential series

$$\sum_{n=0}^{n=\infty} \frac{C}{n!} \left(\frac{2Ml^{2-i}}{1-i}\right)^n.$$

Since this last series is a series with positive constant terms and is known to be convergent, it follows that the series (3) and (4) are both uniformly convergent throughout ab. Each series, therefore, represents a continuous function of x and (4) represents the derivative of (3). We will denote

the value of the series (3) by y. We see then that y is a function which, together with its first derivative, is single valued and continuous throughout ab. To prove that y satisfies (1) at every point of ab other than c let us differentiate (4) term by term

$$-(pY_0' + qY_0) - (pY_1' + qY_1) - (pY_2' + qY_2) - \cdots.$$

It is clear that this same series might have been obtained by multiplying (4) by $-p$ and (3) by $-q$ and adding. It is accordingly uniformly convergent throughout any part of ab which does not contain c, and is therefore equal to y''. On the other hand it is equal to $-py' - qy$. Therefore at every point of ab other than c, y satisfies (1).

Finally when $x = c$ all the terms after the first in both (3) and (4) are zero while the first reduce to γ and γ' respectively. Thus our theorem is established.

We will now prove that *every function y which satisfies* (1) *at all points of the interval $a \leqq x < c$ (or of the interval $c < x \leqq b$) approaches, as does also its derivative y', a finite limit as x approaches c.* To prove this let y_1 and y_2 be two of the solutions just obtained by the method of successive approximations, chosen (and this is evidently possible) so as to be linearly independent of each other. Then $y = C_1y_1 + C_2y_2$ where C_1 and C_2 are two constants. Now since y_1, y_1', y_2, y_2' approach finite limits as x approaches c it follows that the same is true of y and y'.

Finally, we will prove that *if throughout the interval $a \leqq x \leqq b$, y_1 and y_2 are together with their first derivatives single valued and continuous and except when $x = c$ satisfy* (1), *and if $y_1(c) = y_2(c) = \gamma$ and $y_1'(c) = y_2'(c) = \gamma'$, then $y_1 = y_2$ throughout the interval ab.* For consider first the case in which γ and γ' are not both zero. Then if y_1 and y_2 were not identical it is clear that they must be linearly independent. Every other solution y would therefore be expressible in the form: $y = C_1y_1 + C_2y_2$, so that

$$y(c) = (C_1 + C_2)\gamma$$

and

$$y'(c) = (C_1 + C_2)\gamma'.$$

We thus see that $y(c)$ and $y'(c)$ are proportional to γ and γ' whereas we have proved that they can be arbitrarily chosen. This disposes of all cases except that in which $\gamma = \gamma' = 0$, and here the theorem above stated will be established if we can prove that y_1 and y_2 are both identically zero. This, however, follows at once from the case already

considered as otherwise by adding y_1 or y_2 to a solution y_3 which does not vanish (or whose derivative does not vanish) at c we should get a different solution which at c has the same value and the same derivative as y_3, and this we have just proved to be impossible.

The results just obtained can be at once extended to the case in which there is more than one point of discontinuity of p or q in the interval ab. For this purpose it is merely necessary to break up the interval ab into shorter intervals in each of which only one point of discontinuity lies. We thus get the following theorem

I. *If in the interval $a \leqq x \leqq b$, p and q are functions of the kind described in* (A) *there exists one and only one function y which together with its first derivative is single valued and continuous throughout ab, satisfies* (1) *at every point of ab except at the points of discontinuity of p or q, and has at an arbitrarily given point of ab (which may or may not coincide with one of the discontinuities of p or q) an arbitrarily given value while its derivative also has at this point an arbitrarily given value.*

Finally we note that theorems (E) and (F) *Th. of Osc. 1*, pp. 297, 298 hold for the more general case with which we are now dealing in which p and q are functions of the kind described in (A). The same is true of the theorem concerning the separation of the roots of two linearly independent solutions quoted in *Th. of Osc. 2*, p. 366.

In §3 we shall have to deal with the case in which in (1) the coefficient p is still a function of the kind described in (A)* while q has a point of discontinuity c at which it becomes infinite more strongly than the functions described in (A) but only in such a way that the function $(x - c)q$ belongs to the class described in (A). We will here again begin by confining ourselves to the case in which c is the only point of discontinuity of p or q in the interval $a \leqq x \leqq b$, and we will prove that in this case *the method of successive approximations enables us to find a function y_1 which together with its first derivative is single valued and continuous throughout the interval $a \leqq x \leqq b$, which satisfied (1) at every point of this interval, except at c, and which vanishes at c while its derivative has at c the arbitrarily prescribed value γ'.*†

* In fact in the special case with which we there have to deal $p = 0$.

†This is merely a special case of the following theorem which can be proved by the same method: If $p_1, p_2, \cdots, p_k, (x-c)p_{k+1}, (x-c)^2 p_{k+2}, \cdots, (x-c)^{n-k} p_n$ are functions of the kind described in (A) whose only discontinuity is at c, then a function Y exists and can be found by the method above used which with its first $n-1$ derivatives is single valued

In order to prove this we will take as a first approximation

$$Y_0 = \gamma'(x - c),$$

and form the functions $Y_1, Y_2, \cdots$ by means of the formulæ (2), p. 26. The first of the two integrals in (2) is now not obviously convergent since q has at c a discontinuity up to which we cannot in general integrate. In order to prove this integral convergent it is clearly sufficient to prove that

$$Y_n = (x - c)\, \varphi_n(x)$$

where $\varphi_n(x)$ is continuous at c. This formula holds when $n = 0$, for then we have $\varphi_0 = \gamma'$. If then, assuming that it is true for a given value of n, we can prove it true for the next larger value the convergence of our integral will be established. We will then assume the formula true when we give n the value $n - 1$. Then Y_n' is a continuous function of x when $a \leqq x \leqq b$, and the same is therefore true of Y_n. Remembering that $Y_n(c) = 0$ we have by the law of the mean

$$Y_n(x) = (x - c)\, Y_n'(\bar{x})$$

where $|\bar{x} - c| < |x - c|$. We have therefore $\varphi_n(x) = Y_n'(\bar{x})$, and since Y_n' is continuous at c and $Y_n'(c) = 0$ we see that $\varphi_n(x)$ approaches the limit zero as x approaches c. If then we define $\varphi_n(x)$ to have the value zero when $x = c$ we see that φ_n is continuous at c as well as elsewhere.

Having thus justified the definition of the quantities Y_n the next step is to prove that the series (3) and (4), p. 26 are uniformly convergent. For this purpose we need a few inequalities.

In the first place it is clear that we can find a positive quantity M such that at all points of ab (other than c)

$$|p| < M|x - c|^{-i}, \quad |x - c||q| < M|x - c|^{-i}.$$

We then establish by mathematical induction (the reasoning being almost identical with that used in the analogous case on p. 27) the following inequalities

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and continuous throughout ab, which at every point of ab except c satisfies the differential equation:

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = 0,$$

and which at c vanishes together with its first $n - k - 1$ derivatives while the derivatives $\frac{d^{n-k} y}{dx^{n-k}}, \cdots \frac{d^{n-1} y}{dx^{n-1}}$ have arbitrarily prescribed values at c.

$$|Y'_n| < \frac{|\gamma'|}{n!}\left(\frac{2Mt^{1-i}}{1-i}\right)^n,$$

$$|Y_n| < \frac{|\gamma'|}{n!}\left(\frac{2Mt^{1-i}}{1-i}\right)^n t,$$

where as before we have let $|x - c| = t$. By means of these inequalities the uniform convergence of (3) and (4) is established at once as on p. 27, and as there it is shown that (3) therefore represents a function y_1 of the kind desired. Our theorem is thus proved.

A few simple facts about these functions y_1 must be noted before we go further. In the first place it is clear that if $\gamma' = 0$ y_1 is identically zero. Apart from this trivial case it is clear that y_1 *cannot have an infinite number of roots in the interval ab*, for these roots could (by (E) *Th. of Osc. 1*, p. 297) have no other point than c as a limiting point, and this would clearly require the vanishing of $y_1'(c)$.

In the second place *we can write*

$$y_1(x) = (x - c)E(x),$$

where $E(x)$ is single valued and continuous throughout ab and $E(c) = \gamma'$. For $E(x)$ will of course be continuous everywhere in ab except at c. In order to prove our theorem it is therefore sufficient to show that

$$\lim_{x=c} E(x) = \gamma'.$$

This follows at once from the law of the mean which tells us that $y_1(x) = (x - c)\, y_1'(\bar{x})$ where $|\bar{x} - c| < |x - c|$. We have then $E(x) = y_1'(\bar{x})$, so that

$$\lim_{x=c} E(x) = \lim_{\bar{x}=c} y_1'(\bar{x}) = \gamma'.$$

It is clear that the solutions y_1 so far obtained are linearly dependent, differing only in the value of the constant factor γ'. We proceed to prove that *any solution of* (1) *linearly independent of the solutions y_1 just obtained approaches a finite limit different from zero as x approaches c.*

In order to prove this let us remember that if x_0 is a point so near to c that $y_1 \neq 0$ when $|x - c| \leqq |x_0 - c|$

$$y_2 = y_1 \int_{x_0}^{x} \frac{e^{-\int_{x_0}^{x} p\,dx}}{y_1^2}\, dx$$

is a solution of (1) linearly independent of y_1. Writing as above $y_1 = (x - c)\,E(x)$ we have

$$y_2 = (x - c)E(x)\int_{x_0}^{x} \frac{e^{-\int_{x_0}^{x} pdx}}{(x - c)^2[E(x)]^2}dx.$$

Now we will write

$$\frac{1}{[E(x)]^2}e^{-\int_{x_0}^{x} pdx} = K + F(x),$$

where K is a constant different from zero and $F(x)$ is single valued and continuous while $F(c) = 0$. We then get

$$y_2 = -KE(x) + KE(x)\frac{x - c}{x_0 - c} + (x - c)E(x)\int_{x_0}^{x} \frac{F(x)}{(x - c)^2}dx.$$

As x approaches c it is clear that the first of these three terms approaches a finite limit different from zero, while the second approaches zero. By (E) the third also approaches zero. y_2 therefore approaches a limit different from zero. Now every solution of (1) linearly independent of y_1 may be written in the form: $y = C_1y_1 + C_2y_2$ where $C_2 \neq 0$ so that the truth of our theorem is evident.*

The functions y, with which we have been dealing may of course be extended continuously throughout larger intervals where p and q have other discontinuities than c provided that these discontinuities are of the kind described in (A). We are thus led to the following theorem:

II. *If in the interval $a \leqq x \leqq b$, p and $(x - c)q$ (where $a \leqq c \leqq b$) are functions of the kind described in* (A), *there exists one and except for a multiplicative constant only one, function y_1 which together with its first derivative is single valued and continuous throughout ab, satisfies the differential equation* (1) *at every point of ab except at the points of discontinuity of p and q, and vanishes when $x = c$.*

We have so far regarded the coefficients p and q as functions of x only. We shall, however, in what follows be obliged to regard them as involving also a parameter λ.

Let us first suppose that when $a \leqq x \leqq b$, $\lambda_2 \leqq \lambda \leqq \lambda_1$, p and q are functions of the kind described in (B). We begin as before with the case in which there is only one value $x = c$ for which p or q is discontinuous, and apply the method of

* It may be added that the derivatives of solutions linearly independent of y_1 will not in general approach finite limits at c.

successive approximations regarding γ and γ' as constants.* In the analysis of pp. 26–28 we have merely to consider the nature of the dependence upon λ of the quantities which appear. Thus M may be taken as independent of λ. Y_n and Y_n' will be by (C) continuous functions of (x,λ). The analysis moreover proves that (3) and (4) are uniformly convergent not merely with regard to x but with regard to (x,λ) so that y and y' are continuous functions of (x,λ). This result we state as follows:

III. *If when $a \leqq x \leqq b$ and $\lambda_2 \leqq \lambda \leqq \lambda_1$ p and q are functions of the kind described in* (B), *the solution of* (1) *which together with its first derivative is constant for an arbitrarily given value $x = x_0$ $(a \leqq x_0 \leqq b)$, is together with its first derivative a continuous function of (x,λ) when $a \leqq x \leqq b$, $\lambda_2 \leqq \lambda \leqq \lambda_1$.*

Finally we come to the case of differential equations in which p and $(x - c)q$ are functions of the kind considered in (B), concerning which the following theorem may be stated:

IV. *If when $a \leqq x \leqq b$ and $\lambda_2 \leqq \lambda \leqq \lambda_1$, p and $(x - c)q$ (where $a \leqq c \leqq b$) are functions of the kind described in* (B), *the solution of* (1) *which vanishes when $x = c$ and whose derivative has at this point a constant value is when $a \leqq x \leqq b$ and $\lambda_2 \leqq \lambda \leqq \lambda_1$ together with its first derivative a continuous function of (x,λ).*

The proof here is almost precisely the same as for theorem III. The only point which presents any difficulty is the proof that Y_n' is a continuous function of (x,λ). In order to prove this let us write as on p. 30

$$Y_n(x,\lambda) = (x - c)\varphi_n(x,\lambda)$$

so that

$$Y_n'(x,\lambda) = -\int_c^x [pY'_{n-1} + (x - c)q\varphi_{n-1}]\,dx.$$

If then we can prove that the functions φ_n are continuous functions of (x,λ) when $x = c$ (they are obviously continuous for other values of x) we could as before infer that the functions Y_n' are continuous functions of (x,λ). We shall of course here again use the method of mathematical induction: noting that $\varphi_0 = \gamma'$ is continuous, then assuming that $\varphi_1, \varphi_2, \cdots, \varphi_{n-1}$ are continuous. From this it follows that Y_n' is a continuous function of (x,λ). Now we have (see p. 30)

$$\varphi_n(x,\lambda) = Y_n'(\bar{x},\lambda)$$

where $|\bar{x} - c| < |x - c|$, whence the continuity of φ_n when $x = c$ follows at once.

* The result and the method of proof would remain the same if γ and γ' were any continuous functions of λ.

§ 2. *Extension of Results of Previous Papers to Equations with Discontinuous Coefficients for which the Boundary Conditions may Remain General.*

Leaving for the next section equations for which we should have to base our results on Theorems II and IV of the last section, we consider in the present section extensions of our previous results based on Theorems I and III. Here we may state at once the theorem

V. *All the theorems stated in Th. of Osc. 1, §§ 1, 2, and Th. of Osc. 2, § 1, remain true for differential equations whose coefficients instead of being continuous functions of x or (x, λ) are functions of the kind described in* (A) *or* (B).*

The truth of this statement is at once obvious when we refer to the proofs of the theorems in question. These proofs would require only a few modifications which, in view of the results of the last section, are so obvious that it does not seem necessary to dwell upon the matter further.

There remain then to be generalized the results contained in *Th. of Osc. 2*, § 2. We start here from the differential equation (1) in which

$$q(x) = \chi(x) + \psi(x)[C_k x^k + C_{k-1}x^{k-1} + \cdots + C_0],$$

and where p, χ, and ψ are independent of the parameters $C_0, \cdots, C_k$. We have to consider the $k+1$ finite intervals $a_0b_0, a_1b_1, \cdots, a_kb_k$ such that

$$a_0 < b_0 \leqq a_1 < b_1 \leqq a_2 < b_2 \cdots\cdots < b_{k-1} \leqq a_k < b_k,$$

and the $k+1$ points $\xi_0, \xi_1, \cdots, \xi_k$ such that $a_i < \xi_i < b_i$. We will now assume *first* that each of the functions

$$t_i = g_i \int_{\xi_i}^{x} e^{-\int_{\xi_i}^{x} p\,dx}\, dx + f_i \qquad (i = 0, 1, \cdots, k)$$

(where g_i is any constant not zero and f_i any constant) is a continuous function of x throughout the interval $a_i \leqq x \leqq b_i$. This will of course be the case if p is a function of the kind described in (A), but it will also be true in some other cases which we shall find important.† This first assump-

* In fact, in *Th. of Osc. 1*, Theorem VIII, and *Th. of Osc. 2*, Theorem III, it is not necessary that $\phi(x, \lambda)$ should be a function of the kind described in (B) for the whole interval $L > \lambda > l$. It is sufficient that this should be the case for every interval $\lambda_2 \leqq \lambda \leqq \lambda_1$ contained in the interval just mentioned.

†Cf. for instance p. 37.

tion makes the interval $a_i b_i$ of the x-axis correspond in a one to one manner to a finite interval $\sigma_i \tau_i$ of the t_i axis.

Secondly we will assume that

$$\chi e^{2\int_{\xi_i}^{x} p dx} \text{ and } \psi e^{2\int_{\xi_i}^{x} p dx} \qquad (i = 0, 1, \cdots, k)$$

are throughout the interval $\sigma_i \tau_i$ (including the ends) functions of t_i of the kind described in (A) ; and

Thirdly that $\psi(x)$ does not change sign in any one of these intervals or have an infinite number of roots in them.

Under these circumstances we have the following theorem

VI. *There is one and only one real determination of the parameters* $C_0, C_1, \cdots, C_k$ *for which the equation* (1) *has* $k+1$ *solutions* $y_0, y_1, \cdots, y_k$ *such that* y_i $(i = 0, 1, \cdots, k)$ *has just* m_i *roots in the interval* $a_i < x < b_i$ *and satisfies the conditions*

$$\alpha_i' y_i(a_i) - \alpha_i \left[\frac{dy_i}{dt_i}\right]_{x=a_i} = 0,$$

$$\beta_i' y_i(b_i) - \beta_i \left[\frac{dy_i}{dt_i}\right]_{x=b_i} = 0.$$

Here the quantities m_i are any integers positive or zero, and α_i, α_i', β_i, β_i' are any real quantities subject only to the restriction that α_i and α_i' are not both zero and that β_i and β_i' are not both zero.

The proof of this theorem is practically identical with the proof of Theorem IV., *Th. of Osc. 2** so that we will omit it here. It should be noticed that this theorem includes as special cases Theorems IV and V of *Th. of Osc. 2*.

An important application of Theorem VI is to the general linear differential equation of the second order which is everywhere regular

$$\begin{aligned} &\frac{d^2y}{dx^2} + \left(\frac{1 - \varkappa_1' - \varkappa_1''}{x - e_1} + \cdots + \frac{1 - \varkappa_n' - \varkappa_n''}{x - e_n}\right)\frac{dy}{dx} \\ (7) \qquad &+ \frac{1}{f(x)}\left[\frac{\varkappa_1'\varkappa_1'' f'(e_1)}{x - e_1} + \cdots + \frac{\varkappa_n'\varkappa_n'' f'(e_n)}{x - e_n}\right. \\ &\left. + C_{n-2}x^{n-2} + C_{n-3}x^{n-3} + \cdots + C_0\right] y = 0, \end{aligned}$$

where $f(x) = (x - e_1)(x - e_2) \cdots (x - e_n)$.

* The greatest difference comes in in (*a*) *Th. of Osc. 2*, p. 373, where we must now prove ϕ_k to be not a continuous function of (t_k, C_k) but a function of the kind described in (B).

At this point we will define a term which we shall find it convenient to make use of in what follows. *By the fundamental solutions of* (7) *corresponding to the singular point* e_i *we shall understand the solutions of* (7) *which can be developed about* e_i *in the form**

$$y^{(1)} = (x - e_i)^{\kappa_i'}[1 + g_1'(x - e_i) + g_2'(x - e_i)^2 + \cdots],$$
$$y^{(2)} = (x - e_i)^{\kappa_i''}[1 + g_1''(x - e_i) + g_2''(x - e_i)^2 + \cdots].$$

If we have occasion to distinguish between these two solutions we shall do so by speaking of the first as corresponding to the exponent κ_i' the second to κ_i''. Moreover if we recall that a non-singular point c of (7) may be regarded as having exponents 1 and 0 we see that it will be natural to *understand by the fundamental solutions of* (7) *corresponding to a non-singular point* c *the solutions which can be developed about* c *in the forms*

$$y^{(1)} = (x - c) + g_2'(x - c)^2 + g_3'(x - c)^3 + \cdots,$$
$$y^{(2)} = 1 \qquad + g_2''(x - c)^2 + g_3''(x - c)^3 + \cdots.$$

We assume as before that the coefficients of (7) are real for real values of x (Cf. *Th. of Osc. 2*, p. 375 $(a) - (d)$); and letting $k \leqq n - 2$ we consider the $k + 1$ intervals $a_i \leqq x \leqq b_i (i = 0, 1, \cdots, k)$ where

$$a_0 < b_0 \leqq a_1 < b_1 \leqq a_2 < b_2 \cdots\cdots < b_{k-1} \leqq a_k < b_k.$$

If no singular point lies in any of these intervals it is clear as stated in *Th. of Osc. 2*, p. 375, that the simple form of Klein's theorem of oscillation is at once applicable. Let us now enquire how it is if one of the intervals $a_i b_i$ just reaches up with one of its ends to one of the real singular points of (7). Let us say for the sake of distinctness $a_i = e_1$. There are two cases to consider:

1. The case in which the exponents κ_1', κ_1'' are conjugate imaginaries. Here it is at once obvious from the form of development about e_1 that every real solution of (7) has an infinite number of roots in the neighborhood of e_1 so that the theorem of oscillation cannot be extended to this case. We need then merely to consider:

2. The case in which κ_1', κ_1'' are real. We will at first consider here only the special case (to which, however, as

* In case $\kappa_i' - \kappa_i''$ is an integer, that one of these series into which the smaller of the exponents enters would have to be replaced by a more complicated expression which, however, we shall not have occasion to use.

we shall see presently the general case can be reduced*) in which $\varkappa_1'' = 0$ while $\varkappa_1' > 0$. Here

$$t_i = g_i \int_{\xi_i}^{x} e^{-\int_{\xi_i}^{x}\left[\frac{1-\kappa_1'}{x-e_1}+\frac{1-\kappa_2'-\kappa_2''}{x-e_2}+\cdots+\frac{1-\kappa_n'-\kappa_n''}{x-e_n}\right]dx}\, dx + f_i$$

$$= (x-e_1)^{\kappa_1'} E(x) + \sigma_i,$$

where $E(x)$ is continuous throughout the interval $e_1 \leqq x \leqq b_i$ and $E(e_1) \neq 0$. The condition of continuity of t_i is therefore satisfied since by hypothesis $\varkappa_1' > 0$.

A relation which we shall need in a moment can at once be deduced from the value of t_i just obtained, viz:

$$x - e_1 = (t_i - \sigma_i)^{\frac{1}{\kappa_1'}} E_1(t_i)$$

where E_1 is a single valued and continuous function of t_i throughout the interval $\sigma_i \tau_i$ including the ends, and

$$E_1(\sigma_i) \neq 0.$$

Going a step further we have

$$\chi(x) = \frac{1}{f(x)}\left[\frac{\varkappa_2'\varkappa_2'' f'(e_2)}{x-e_2} + \cdots + \frac{\varkappa_n'\varkappa_n'' f'(e_n)}{x-e_n} + C_{n-2}x^{n-2} + \cdots + C_{k+1}x^{k+1}\right],$$

$$\psi(x) = \frac{1}{f(x)},$$

where $f(x) = (x-e_1)(x-e_2)\cdots(x-e_n)$, and finally

$$e^{2\int_{\xi_i}^{x} p\,dx} = k(x-e_1)^{2-2\kappa_1'}(x-e_2)^{2-2\kappa_2'-2\kappa_2''}\cdots(x-e_n)^{2-2\kappa_n'-2\kappa_n''}$$

where k is a constant different from zero. We may therefore write, using the value of $x - e_1$ in terms of t_i above obtained

$$\chi(x) e^{2\int_{\xi_i}^{x} p\,dx} = (x-e_1)^{1-2\kappa_1'} F_1(x) = (t_i - \sigma_i)^{\frac{1}{\kappa_1'}-2}\, \Phi_1(t_i),$$

$$\psi(x) e^{2\int_{\xi_i}^{x} p\,dx} = (x-e_1)^{1-2\kappa_1'} F_2(x) = (t_i - \sigma_i)^{\frac{1}{\kappa_1'}-2}\, \Phi_2(t_i),$$

* Except the case of equal exponents which as we shall see is excluded.

where F_1 and F_2 are both continuous throughout the interval $e_1 \leqq x \leqq b_i$, and Φ_1 and Φ_2 are both continuous throughout the interval $\sigma_i \tau_i$ including the ends. The functions

$$\chi(x) \cdot e^{2\int_{\xi_i}^{x} p dx} \quad \text{and} \quad \psi(x) \cdot e^{2\int_{\xi}^{x} p dx}$$

are therefore functions of t_i of the kind described in (A) provided that $0 < \varkappa_1' < 1$. *If this last inequality holds theorem VI may therefore be applied.* Moreover it is evident that instead of allowing one of the points $a_0, b_0, a_1, b_1, \cdots$ to coincide with a singular point of (7) we might have allowed any number of them to do so provided that one of the exponents of each of the singular points thus involved is zero while the other is real, greater than zero, and less than one.

The conditions to which the solutions y_i are subjected at the points $a_i b_i$ may be stated in the cases with which we are now concerned in a very simple form if we use the conception of the fundamental solutions corresponding to these points. Suppose for instance that the singular point e_1 is one of the points a_i, then y_i satisfies the condition

$$\alpha_i' y_i(e_1) - \alpha_i \left(\frac{dy_i}{dt_i}\right)_{x=e_1} = 0.$$

Now y_i can be written $k^{(1)}y^{(1)} + k^{(2)}y^{(2)}$ where $k^{(1)}$ and $k^{(2)}$ are constants and $y^{(1)}$ and $y^{(2)}$ are the fundamental solutions corresponding to the exponents $\varkappa_1'$ and 0 respectively at e_1. Substituting this value of y_i into the above condition we readily see, when we use the explicit expression for t_i in terms of x, that the condition reduces to

$$\alpha_i' k^{(2)} - \alpha_i K k^{(1)} = 0$$

where K is a constant different from zero and independent of the parameters C. Now since α_i' and α_i can be chosen as any two constants not both zero we see at once that *the condition can be expressed by saying that y_i is to be proportional to an arbitrarily chosen linear combination of the fundamental solutions corresponding to e_1.*

The condition $\gamma' y(c) - \gamma y'(c) = 0$ at a non-singular point may also be stated in this form, for letting $y = k^{(1)}y^{(1)} + k^{(2)}y^{(2)}$, where $y^{(1)}$ and $y^{(2)}$ are the fundamental solutions corresponding to c, we see at once that our condition reduces to $\gamma' k^{(2)} - \gamma k^{(1)} = 0$, and this is equivalent to requiring that y be proportional to an arbitrarily chosen linear combination of $y^{(1)}$ and $y^{(2)}$.

So far we have merely considered the case in which one of the exponents of each of the singular points with which we have to deal is zero. Let us now consider the more general case in which the exponents $\varkappa_i'$ and $\varkappa_i''$ of these points are any real quantities where $\varkappa_i' > \varkappa_i''$. We will let $\varkappa_i' - \varkappa_i'' = \varkappa_i$; and denote by $\Pi(x - e_i)^{\varkappa_i''}$ the product of the expressions $(x - e_i)^{\varkappa_i''}$ taken for all the singular points which coincide with any of the points a_i or b_i. Now let

$$y = \Pi(x - e_i)^{\varkappa_i''} \cdot \bar{y}$$

and we get as the differential equation satisfied by $\bar{y}$ an equation differing from (7) in only two respects:

1st, the exponents of the points e_i which coincide with points a_i or b_i are now $\varkappa_i$ and 0 instead of $\varkappa_i'$ and $\varkappa_i''$.

2d, the parameters $\bar{C}_0, \bar{C}_1, \cdots \bar{C}_{n-2}$ of the transformed equation are connected with the parameters $C_0, C_1, \cdots, C_{n-2}$ of the original equation by relations of the form: $\bar{C}_i = C_i + k_i$ where the constants k_i are independent of the C's.

We may therefore determine the parameter $\bar{C}$ by applying the theorem of oscillation to the transformed equation provided that $0 < \varkappa_i < 1$; and the $\bar{C}$'s being thus determined the C's are uniquely determined. It remains then merely to show that we can so arrange the conditions of the theorem of oscillation when we apply it to the transformed equation as to give any desired conditions for the original equation.

In the first place it is clear that the roots of a solution y_i in an interval $a_i < x < b_i$ are precisely equal to those of the corresponding solution $\bar{y}_i$ in this interval, and therefore in particular that y_i and $\bar{y}_i$ have the same number of roots in this interval. Moreover if $y^{(1)}$ and $y^{(2)}$ are the fundamental solutions of (7) corresponding to a point c (whether this point be singular or non-singular is immaterial) $\bar{y}^{(1)}$ and $\bar{y}^{(2)}$ although not themselves the fundamental solutions of the transformed equation differ from them only by the same constant factor. We may then state the following theorem:

VII. *If the coefficients of equation* (7) *are real for real values of x and if the $k + 1$ intervals* $(k \leqq n - 2)$ $a_0b_0, a_1b_1, \cdots, a_kb_k$ *are so situated that*

1st, $a_0 < b_0 \leqq a_1 < b_1 \leqq a_2 < b_2 \cdots\cdots < b_{k-1} \leqq a_k < b_k$;

2d, *no singular point of* (7) *lies in any of the intervals*

$$a_i < x < b_i (i = 0, 1, \cdots, k);$$

3d, *the points a_i and b_i themselves, so far as they are singular points of* (7), *have unequal real exponents whose difference is numerically less than* 1;

Then, the quantities m_i being any integers positive or zero, there is one and only one real determination of the parameters $C_0, C_1, \cdots, C_k$ for which (7) *has $k+1$ solutions $y_0, y_1, \cdots, y_k$ such that y_i has just m_i roots in the interval $a_i < x < b_i$ and at each extremity of this interval is proportional to an arbitrarily chosen linear combination of the fundamental solutions corresponding to this extremity.*

§ 3. *Extension of Results of Previous Papers to Equations with Discontinuous Coefficients for which the Boundary Conditions must be Specialized.*

We will begin here by considering equations of the form:

$$\frac{d^2y}{dx^2} = \varphi \cdot y \tag{8}$$

where the coefficient φ is such that throughout the interval $a \leqq x \leqq b$, $(x-a)\varphi$ is a function of the kind described in (A) or (B). Here we see by Theorems II and IV that we can no longer require of solutions y that they shall satisfy the conditions $y(a) = \alpha$, $y'(a) = \alpha'$ where α and α' are arbitrarily given, but that we must now restrict ourselves to the boundary condition $y(a) = 0$. With this restriction, however, many of the results of the previous papers may be extended to the case now before us by methods practically identical with those there given. We may say:

VIII. *All the theorems contained in Th. of Osc. 1, § 1, 2 as well as Theorem III, Th. of Osc. 2 remain true if instead of φ being a continuous function of x or (x, λ), $(x-a)\varphi$ is a function of the kind described in* (A) *or* (B), *provided we let $\alpha = 0$ in these theorems.*

Theorems almost identical with these would of course hold if the points a and b were interchanged (Cf. *Th. of Osc. 1*, § 2, last paragraph).

The methods used in the previous papers are not, however, applicable to the case in which φ becomes infinite both at a and at b more strongly than the functions described in (A) or (B). We shall, therefore, need a somewhat new method to prove the following theorem of oscillation:

IX. *If when $a \leqq x \leqq b$ and $L > \lambda > l$,* $(x-a)(x-b)\varphi(x, \lambda)$ is a function of the kind described in* (B), *and if when $a \leqq x \leqq b$*

* We may have $L = +\infty$ or $l = -\infty$ or both L and l may be infinite. Cf. also the first footnote on p. 34 which applies here.

and $L > \lambda' > \lambda'' > l$, $\varphi(x, \lambda') \geqq \varphi(x, \lambda'')$;† *if, moreover*

$$\lim_{\lambda=L} \varphi(x, \lambda) = +\infty \text{ and } \lim_{\lambda=l} \varphi(x, \lambda) = -\infty$$

except perhaps for a finite number of values of x *in the interval* $a \leqq x \leqq b$ *; then there will be one and only one value of* λ *in the interval* $L > \lambda > l$ *for which* (8) *has a solution which has just* m *roots* (m *being any integer positive or zero*) *in the interval* $a < x < b$ *and which vanishes when* $x = a$ *and when* $x = b$.

In order to prove this let us take a point c so that $a < c < b$. Let us denote by y_1 and y_2 solutions of (8) which vanish at a and b respectively. It will be convenient to speak in the following proof of the *aggregate number of roots of* y_1 *and* y_2 *in* ab meaning thereby the integer obtained by adding to the number of roots of y_1 in the interval $a < x < c$ the number of roots of y_2 in the interval $c < x < b$ and increasing this number by 1 if y_1 or y_2 (or both) vanish at c. Our theorem will clearly be proved if we can show that there is one and only one value of λ in the interval $L > \lambda > l$ for which y_1 and y_2 are linearly dependent while this aggregate is m. For if y_1 and y_2 are linearly dependent they can be made equal by multiplying one of them by a suitable constant.

Now we see from VIII (and *Th. of Osc. 2*, theorem III) that there is one and only one value of λ in the interval $L > \lambda > l$, say λ_{m_1}', for which $y_1(x, \lambda_{m_1}')$ has m_1 roots in the interval $a < x < c$ while $y_1(c, \lambda_{m_1}') = 0$. There is also one and only one value of λ in the interval $L > \lambda > l$, say λ_{m_2}'', for which $y_2(x, \lambda_{m_2}'')$ has m_2 roots in the interval $c < x < b$ while $y_2(c, \lambda_{m_2}'') = 0$. Let us consider the two sequences $\lambda_0', \lambda_1', \lambda_2', \cdots$, and $\lambda_0'', \lambda_1'', \lambda_2'', \cdots$. Each of these sequences is arranged in order of decreasing magnitude. Let us now arrange the terms of the two sequences in a single sequence $\lambda_0, \lambda_1, \lambda_2, \cdots$, in order of decreasing magnitude ; it being understood that if $\lambda_i' = \lambda_j''$ these two terms shall be represented by only one term in the new sequence. Let us denote by μ_k the aggregate number of roots of $y_1(x, \lambda_k)$ and $y_2(x, \lambda_k)$ in ab. Then it is clear that in general $\mu_{k+1} = \mu_k + 1$, the only exception being that if $y_1(c, \lambda_k) = y_2(c, \lambda_k) = 0$, $\mu_{k+1} = \mu_k + 2$. In the sequence of integers $\mu_0, \mu_1, \cdots$, one at least of the two integers m and $m - 1$ must, therefore, be present. Let us

† This inequality need not of course hold for the points of discontinuity of φ. Moreover, there must be some points (besides these poinst of discontinuity) where the equality sign does not hold.

give to k the value which makes $\mu_k = m$ if this is possible, and if not the value which makes $\mu_k = m - 1$.

One case we can dispose of at once, viz., that in which $\mu_k = m$ and $y_1(c, \lambda_k) = y_2(c, \lambda_k) = 0$. When $\lambda > \lambda_k$ the aggregate number of roots of y_1 and y_2 in ab is less than m, when $\lambda < \lambda_k$ it is greater than m. Accordingly when $\lambda = \lambda_k$ and for this value only is the aggregate equal to m. Moreover, for this value y_1 and y_2 are linearly dependent since they both vanish when $x = c$. In this case then our theorem is proved.

Apart from this case, which we now exclude, it is clear that $\lambda = \lambda_k$ is not the value of λ we are seeking. Leaving aside this value then we see that the aggregate number of roots of y_1 and y_2 in ab is equal to m when and only when $\lambda_k > \lambda > \lambda_{k+1}$. It remains then to prove that in this range of values of λ there is one and only one value for which y_1 and y_2 are linearly dependent, *i. e.* (since $y_1(c, \lambda) \neq 0$ and $y_2(c, \lambda) \neq 0$) for which $y_1'(c, \lambda)/y_1(c, \lambda) = y_2'(c, \lambda)/y_2(c, \lambda)$. This follows at once from *Th. of Osc. 1*, theorem II for as λ decreases from λ_k to λ_{k+1} $y_1'(c, \lambda)/y_1(c, \lambda)$ will be decreasing continuously while $y_2'(c, \lambda)/y_2(c, \lambda)$ is increasing continuously. Moreover we must have either $y_1(c, \lambda_k) = 0$ or $y_2(c, \lambda_k) = 0$ and either $y_1(c, \lambda_{k+1}) = 0$ or $y_2(c, \lambda_{k+1}) = 0$. Accordingly when $\lambda = \lambda_k$ and also when $\lambda = \lambda_{k+1}$ one of the two ratios y'/y just considered is infinite. There are four possibilities here, viz.: 1st, y_1'/y_1 decreases from $+\infty$ to $-\infty$; 2d, y_1'/y_1 decreases from $+\infty$ to a finite limit while y_2'/y_2 increases from a finite limit to $+\infty$; 3d y_1'/y_1, decreases from a finite limit to $-\infty$ while y_2'/y_2 increases from $-\infty$ to a finite limit; 4th, y_2'/y_2 increases from $-\infty$ to $+\infty$. In any case, however, it is clear that the two ratios become equal once and only once, as was to be proved.

In this proof we have tacitly assumed that $m \neq 0$. The changes necessary in the case $m = 0$ are however so slight and so obvious that it does not seem necessary to dwell upon them.*

The generalization of Klein's theorem of oscillation follows now so readily that we merely give the result:

X. *Theorem* VI *still holds if we drop at some or all of the points* a_i *and* $b_i (i = 0, 1, \cdots, k)$ *the requirements that*

$$\chi \cdot e^{2\int_{\xi_i}^{x} p dx} \quad \textit{and} \quad \psi \cdot e^{2\int_{\xi_i}^{x} p dx}$$

* Were it not for this case $m = 0$ we might have followed a slightly simpler method analogous to that used by Picard in a somewhat similar case. Cf. Traité d'Analyse, vol. 3, p. 122.

be functions of the kind described in (A) *and merely require that these functions when multiplied by* $x - a_i$ (*or by* $x - b_i$) *are functions of the kind deseribed in* (A) PROVIDED, *that at the points* a_i *and* b_i *just mentioned we require that* $\alpha_i = 0$ *and* $\beta_i = 0$.

If we apply this theorem to equation (7) we get

XI. *Theorem* VII *still holds if we allow some or all of the points* a_i *and* b_i *to be singular points with unequal real exponents whose difference is no longer restricted to being numerically less than* 1, PROVIDED *that at each of these points we require that the corresponding solution* y_i *should be proportional to the fundamental solution corresponding to the greater exponent of this point.*

Theorems VII and XI include all the cases for which Klein has discussed the theorem of oscillation in his lithographed lectures: Lineare Differentialgleichungen der zweiten Ordnung, 1894, or for which he has there expressed the opinion, based on analogy with simpler cases, that the theorem is probably true.

We close by noting two slight generalizations:

1st The proof of theorems VII and XI would not be in the least affected if instead of the polynomial

$$C_{n-2}x^{n-2} + \cdots + C_0$$

which enters equation (7) we had a polynomial of degree greater than $n - 2$, in which case the point $x = \infty$ would be an irregular point. The number $k + 1$ of intervals $a_i\, b_i$ might of course then be correspondingly increased.

2d Theorems VII and XI still remain true, the proofs not being essentially affected, if several of the points $e_1, e_2, \cdots, e_n$ coincide, provided that none of the intervals $a_i\, b_i$ reaches up to such a "multiple" point, which will obviously in general be an irregular point.

[*Note added Sept. 14, 1898:* It was not until the present paper was in print that I noticed that some of the theorems contained in it had been proved for a particular differential equation and in somewhat restricted form in a paper by Van Vleck "On the Polynomials of Stieltjes" in the June number of the BULLETIN. There is even an analogy of method between the proof of Van Vleck's Theorem IV and that of my Theorem IX. It may therefore be well to state that I was in possession of all the results and methods of the present paper last February (Cf. a remark in the middle of p. 365 *Th. of Osc. 2*) and was prevented from working them out in detail only by press of other work.]

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