representing $-\varphi(x)$ within this circle all of Poincaré's analysis applies without modification. Hence this circle is the true circle of convergence for this series.

Finally, for the case that x_0 is any point of A, Poincaré's reasoning, with the modification just given, still holds, and the theorem is thus established that $\varphi(x)$ is analytic in A, but cannot be continued beyond A.

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SUPPLEMENTARY NOTE ON A SINGLE-VALUED FUNCTION WITH A NATURAL BOUNDARY, WHOSE INVERSE IS ALSO SINGLE-VALUED.

BY PROFESSOR W. F. OSGOOD.

(Read before the American Mathematical Society at its Fifth Summer Meeting, Boston, Mass., August 19, 1898.)

IN the June number of the BULLETIN I gave an example of a single-valued function with a natural boundary, the inverse of which is also single-valued. The function employed was the following:

$$f(z) = z + \frac{z^{a+2}}{(a+1)(a+2)} + \frac{z^{a^2+2}}{(a^2+1)(a^2+2)} + \cdots,$$

where a is a positive integer greater than unity. This function is continuous within and on the boundary of the unit circle, is analytic within this circle, and cannot be continued analytically beyond it.

I am indebted to Professor Hurwitz for an exceedingly simple proof of the principal theorem of my note, namely, that the inverse function is single-valued. The point to be established is that, z, z' being any two distinct points within or on the unit circle,

$$f(z) \neq f(z').$$

This follows at once by the application of a method employed by Professor Fredholm* to show that the inverse of the function

* *Cf.* Verhandlungen des ersten internationalen Mathematiker-Kongresses in Zürich vom 9. bis 11. August 1897; herausgegeben von Dr. Ferdinand Rudio, Professor am eidgenössischen Polytechnikum; Teubner, 1898; p. 109.

$$F(z) = \sum_{n=0}^{\infty} a^n z^{n^2}, \qquad (|a| < 1),$$

is single-valued, provided $|a|$ is not too large.

The proof is as follows. Evidently

$$\left|\frac{f(z) - f(z')}{z - z'}\right| = \left|1 + \sum_{n=1}^{\infty} \frac{z^{a^n+1} + z^{a^n} z' + \cdots + z'^{a^n+1}}{(a^n+1)(a^n+2)}\right|$$

$$\geqq 1 - \sum_{n=1}^{\infty} \frac{|z|^{a^n+1} + |z|^{a^n}|z'| + \cdots + |z'|^{a^n+1}}{(a^n+1)(a^n+2)}$$

$$\geqq 1 - \sum_{n=1}^{\infty} \frac{1}{a^n+1} = 1 - \frac{1}{a+1} - \left(\frac{1}{a^2+1} + \frac{1}{a^3+1} + \cdots\right)$$

$$> 1 - \frac{1}{a+1} - \frac{1}{a(a-1)} > 0.$$

Hence $$|f(z) - f(z')| > 0,$$ q. e. d.

HARVARD UNIVERSITY, CAMBRIDGE, MASS.

NOTE ON THE PERIODIC DEVELOPMENTS OF THE EQUATION OF THE CENTER AND OF THE LOGARITHM OF THE RADIUS VECTOR.

BY PROFESSOR ALEXANDER S. CHESSIN.

IF we put with Professor S. Newcomb*

(1) $$E = ev_1 + e^2 v_2 + e^3 v_3 + \cdots$$

(2) $$\rho - \log a = e\rho_1 + e^2\rho_2 + e^3\rho_3 + \cdots$$

where E stands for the equation of the center and $\rho = \log r$, then v_i and ρ_i will be of the form

(3) $$iv_i = \tfrac{1}{2}\Sigma k_j^{(i)} \sin j\zeta,$$

(4) $$i\rho_i = \tfrac{1}{2}\Sigma h_j^{(i)} \cos j\zeta,$$

$$(j = i, \quad i-2, \quad i-4, \quad \cdots \quad , -i),$$

* "Development of the perturbative function," Astronomical Papers prepared for the use of the American Ephemeris and Nautical Almanac, vol. 5, part I., p. 12.