SCHELL'S TORTUOUS CURVES.

Allgemeine Theorie der Curven doppelter Krümmung in rein geometrischer Darstellung. Zur Einführung in das Studium der Curventheorie. Von DR. WILHELM SCHELL. Zweite, erweiterte Auflage. Leipzig, Teubner, 1898. 8vo, viii + 163 pp.

The first edition of Dr. Schell's Theory of tortuous curves appeared in 1859. What distinguishes this work from other treatises on the subject and makes it particularly valuable as a first introduction to the study of curves of double curvature is the simple and natural, strictly geometrical, method used throughout. The curve is here studied in itself, with the aid of the various lines and surfaces naturally connected with it, but without any reference to a system of coördinates in the ordinary sense. Barré de Saint-Venant seems to have been the first to appreciate the great power and simplicity of such a method, as appears from the third part of his classical memoir "Sur les lignes courbes non planes" (*Journal de l'Ecole polytechnique*, vol. 18 (1845), 30^e cahier, pp. 1–76). But while the opening remarks of this memoir show that de Saint-Venant attached no small importance to this method, it is still true that in his presentation of the theory of curves the purely geometrical treatment comes in as a kind of afterthought, as if intended merely as a means of verifying subsequently the complicated analytical calculations of the earlier parts of the paper. Besides, in de Saint-Venant's memoir the method is still far from receiving its full development.

Professor Schell's work has been known so long and so favorably that it will be sufficient to call attention to some of the additions and modifications introduced in the new edition. While the general plan of the work has remained the same, the changes are so extensive that the book has grown from 106 to 163 pages; it is revised throughout and carefully brought up to date. The insertion of headings to the sections facilitates its use for reference; but the blessing of an alphabetical index is still missing.

The introduction is almost entirely rewritten. In the first chapter of the book, after a general survey over the lines and surfaces most closely related to a tortuous curve (tangent, principal normal, binormal, rectifying line, polar axis, and the surfaces generated by these lines), the discussion of singular points, which has attracted some attention in recent years, is now illustrated by a number of figures showing the most important singularities possible for a tortuous curve according to von Staudt's classification. While the treatment of this subject is still rather brief, the references given on p. 16 to the works of Fiedler and Wiener and to papers by Kneser, Fine, Björling, Staude, will enable the student to pursue the investigation farther.

The five following chapters (Chap. 2, on the three curvatures and their radii; Chap. 3, on the surface generated by the tangents and on the involutes of the curve; Chap. 4, on the surface generated by the polar axis and on the evolutes of the curve; Chap. 5, on the osculating sphere and cone; Chap. 6, on the surface enveloped by the rectifying plane) are not very essentially changed, although the revising hand is discernible almost on every page. The more important additions consist in the introduction of Chasles's terms "central point," "central plane" and "distributing parameter," of a generator of a scroll, in the fuller discussion of the geometrical interpretation of the radii of torsion and of total curvature, in some additional propositions about the edges of regression of the surface of the polar axes (p. 62), the rectifying surface and the surface of evolutes (p. 66–67), in the discussion of the normal sections of the rectifying surface (p. 83) and of the problem of determining curves from a given relation between their radii of curvature and torsion (pp. 83–85). The further development of the last subject, which is excluded from this work, would lead to what Professor Cesàro calls "intrinsic," and Professor Bianchi "natural," geometry of curves.

De Saint-Venant's problem concerning curves having their principal normals in common has led to so many new investigations that the next two chapters (Chap. 7, the scroll generated by the principal normals; Chap. 8, curves of constant curvature and curves with a common scroll of principal normals) had to be considerably enlarged to embody the results of the investigations of J. Bertrand, J. A. Serret, Mannheim, Fais, and others. As a tortuous curve forms, on the surface of its principal normals, an asymptotic line cutting these normals at right angles, the problem of determining how many curves on a given scroll can have this surface as the common surface of their principal normals is equivalent to the question of determining how many asymptotic lines can intersect a generator g of the scroll at right angles. This form seems to have been first given to the problem by Professor Mannheim (1872). The solution is almost immediate because the scroll can, for the present purpose, be replaced by its osculating hyperboloid, *i. e.*, by the hyperboloid determined by three successive generators g, g', g'' of the scroll. The asymptotic cone of this hyperboloid being of the second order, a plane perpendicular to g through its vertex will intersect the cone in 0, 1, or 2 lines; and those generators of the paraboloid which are parallel to these lines of the cone are evidently tangent to the only curves that can have g as a common normal at their points of intersection with g. Hence, a given scroll may be the surface of principal normals for no curves, or for one curve, or for two curves. In the particular case when the osculating hyperboloid degenerates into a hyperbolic paraboloid the scroll will contain an infinite number of such curves; the scroll is in this case a common helicoid. The interesting properties of two curves having a common scroll of principal normals are proved by Professor Schell in a very simple way. Starting with Bonnet's proposition that the osculating planes of the curves at two corresponding points form a constant angle, the linear relation shown by Bertrand to exist between the curvature and torsion of either curve is deduced, the constancy of the distance between corresponding points is proved, and expressions are derived for the radii of curvature and torsion of one of the curves in terms of those of the other.

The scroll of binormals and the surface enveloped by the planes of total curvature are briefly discussed in the ninth chapter. Chapter 10, originally devoted only to the osculating helix, is enriched by the addition of an account of a conical loxodrome having with the curve a contact of the third order (pp. 122–127).

The whole of chapter 11 (pp. 128–141) is new; it applies the theory of the infinitesimal displacement of a rigid body to the trirectangular trihedral formed by the tangent, principal normal, and binormal at any point of the curve (Bianchi's principal trihedral). After deducing the indispensable lemmas from the geometry of motion here required, the author shows that the infinitesimal displacement of the trihedral, as its vertex describes an element of the curve, is a twist (or screw-motion) whose axis is the shortest distance of two successive principal normals, *i. e.*, the axis of the osculating helix, while the angle of rotation is the angle of total curvature. The translation and pitch of the twist are easily determined. A brief discussion is also given of the two ruled surfaces (sometimes called "axoids") formed by the successive positions, in space and in the trihedral, of the instantaneous axes of twist; and it is shown that the surface connected with the trihedral becomes a cylindroid if the radius of torsion of the curve is constant.

The last two chapters of the book, on cyclifying surfaces and evolutoids, have not undergone much change.

The work seems to be unusually free from serious misprints or inaccuracies. The following corrections might perhaps be worth mentioning: p. 16, footnote: Dr. Kneser's first paper appeared in Vol. XXXI (not XXI). the second will be found in Vol. XXXIV, at p. 204 (not XXIV, p. 506), of the *Mathematische Annalen*; p. 61, end of §2: the last expression for tan μ is obviously wrong, the last but one should have in the denominator ds instead of $d\sigma$; p. 91: the proof of the relation $A' = B + (\frac{1}{2}\pi - \alpha)$ seems unduly long, as the triangle APB (Fig. 34), all of whose sides are infinitesimal of the same order while P is a right angle, gives at once $\frac{1}{2}\pi + B$ for the exterior angle $\alpha + A'$ at A; p. 110, l. 11 from foot of page: read $= 1$ for $- 1$, and in the numerator of a read $-1/r$ for $1/r$; p. 137, l. 15 from top: for Hauptnormalen read Binormalen.

The concluding remarks of the work give an interesting outlook on problems awaiting solution in the geometrical theory of tortuous curves. Coming as they do from one who has made a special study of the subject they will be read with great interest by all workers in this field.

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PAGE'S DIFFERENTIAL EQUATIONS.

Ordinary Differential Equations. An elementary text-book, with an introduction to Lie's theory of the group of one parameter. By JAMES MORRIS PAGE, PH.D., Adjunct Professor of Mathematics, University of Virginia. The Macmillan Company, New York, 1897. 12mo, xviii + 226 pp.

This little volume is what it purports to be,—an elementary text-book with an introduction to Lie's elementary methods of integration as applied to ordinary differential equations. The contents fall into twelve chapters devoted to the following subjects in order: I. Genesis of the ordinary differential equation in two variables, pp. 1–9; II. The simultaneous system and the equivalent linear partial differential equation, pp. 10–24; III. The fundamental theorems of Lie's theory of the group of one parameter, pp. 25–59; IV. Connection between Euler's integrating factor and Lie's infinitesimal transformation, pp. 62–97; V. Geometrical