THE ROOTS OF POLYNOMIALS WHICH SATISFY CERTAIN LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

BY PROFESSOR MAXIME BÔCHER.

(Read before the American Mathematical Society at the Meeting of December 29, 1897.)

IN volume 6 of the *Acta Mathematica*, Stieltjes has given a remarkable method for showing in how many different ways certain parameters in an important class of linear differential equations of the second order can be so determined that the equation shall have a polynomial solution, and in the course of the work the position of the roots of these polynomials is determined. It sometimes happens that the answer to the first part of the problem here referred to can be obtained more easily or more naturally by other methods. For instance in the case of the hypergeometric differential equation the forms in which solutions can be expanded in series show us at once in what cases we have a polynomial solution; and in the case of Lamé's equation the theorem of oscillation leads us most naturally (from some points of view) to the result.* There still remains the second part of the above problem, viz.: the determination of the position of the roots of the polynomials. The method of Stieltjes is connected with a problem in the equilibrium of particles on a straight line. By generalizing these considerations so as to bring in particles lying in a plane, we can, as I have shown,† obtain a theorem concerning the position of the roots of the polynomials, which, though in itself less far reaching (in some respects) than that of Stieltjes, gives us in the cases above referred to the information we want. I should like here to emphasize three points:

1. This method enables us to avoid the determination of an upper limit to the number of determinations of the parameters which give polynomial solutions.

2. It enables us to go beyond the cases considered by Stieltjes inasmuch as the singular points of the differential equation may now be complex.

3. Owing to the relatively small result we wish to attain

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* Cf. mv book: Ueber die Reihenentwickelungen der Potentialtheorie, Leipzig, Teubner, 1894. See pages 210–213.

† See pages 215–216 of the book just referred to.

it is possible to throw the proof into purely *algebraic* form; whereas Stieltjes's method involving, as it does, an existence-proof depends upon transcendental considerations.

I will now state the theorem and give its proof in the algebraic form to which I have referred, mentioning however, for the sake of brevity, only the case in which the singular points of the differential equation are real.

Let $\varphi(x) = (x - x_1)(x - x_2) \cdots (x - x_k)$ *be a polynomial which satisfies the differential equation:*

$$\frac{d^2y}{dx^2} + \left(\frac{a_1}{x - e_1} + \frac{a_2}{x - e_2} + \cdots + \frac{a_n}{x - e_n}\right)\frac{dy}{dx} + \frac{\psi(x)}{(x - e_1)(x - e_2)\cdots(x - e_n)}y = 0,$$

in which $\psi(x)$ *is a polynomial. Let* $e_1, e_2, \cdots, e_n$ *be real and unequal* $(e_1 < e_2 \cdots < e_n)$. *Then if* $a_1, a_2, \cdots, a_n$ *are real and greater than zero the roots* $x_1, x_2, \cdots, x_k$ *of* $\varphi(x)$ *must all be real and must lie in the interval* $e_1 \leqq x \leqq e_n$.

It is evident at once from the elements of the theory of multiple roots that $\varphi(x)$ can have no multiple root which is not equal to one of the quantities $e_1, e_2, \cdots, e_n$. Let x_1 be any root of $\varphi(x)$ different from $e_1, e_2, \cdots, e_n$. Then if we substitute $\varphi(x)$ in the differential equation and let $x = x_1$ we get:

$$\varphi''(x_1) + \left(\frac{a_1}{x_1 - e_1} + \frac{a_2}{x_1 - e_2} + \cdots + \frac{a_n}{x_1 - e_n}\right)\varphi'(x_1) = 0;$$

or dividing by $\varphi'(x_1)$ (which is not zero since x_1 is not a multiple root):

$$\frac{2}{x_1 - x_2} + \frac{2}{x_1 - x_3} + \cdots + \frac{2}{x_1 - x_k} + \frac{a_1}{x_1 - e_1} + \frac{a_2}{x_1 - e_2} + \cdots + \frac{a_n}{x_1 - e_n} = 0.*$$

Now if $\varphi(x)$ has complex roots with positive pure imaginary part let x_1 be that one (or one of those) whose pure imaginary part is greatest. Then the above equation involves a contradiction for the pure imaginary part of each term is negative or zero, and not all of them are zero since the a's are not zero.

* The first member of this equation is a quantity (perhaps complex) whose conjugate gives both in magnitude and in direction the force acting upon the particle x_1 in the mechanical problem just referred to. The equation therefore gives the condition of equilibrium. The mechanical meaning of the following proof is obvious.

In a similar way we see that φ (x) can have no complex root whose pure imaginary part is negative.

$x_1, x_2, \cdots, x_k$ are, therefore, all real. Suppose one of them were greater than e_n. Call this one (or, if there are more than one, the greatest of them) x_1. Then the above equation again involves a contradiction since no term is negative or zero.

In the same way we see that no root can be less than e_1.

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INFLEXIONAL LINES, TRIPLETS, AND TRIANGLES ASSOCIATED WITH THE PLANE CUBIC CURVE.

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(Read before the American Mathematical Society at the Meeting of February 26, 1898.)

THE configuration of the nine inflexions of a nonsingular plane cubic and the twelve lines containing them three-and-three would seem too well known to merit discussion. It is the uniform mode, in such compends as I have seen, to show first that every line joining two inflexions meets the cubic again in a third inflexion; second, that through the nine inflexions there must lie in all twelve such lines; and thirdly, that three lines can be selected which contain all nine inflexions. These three lines are termed an inflexional triangle, and the entire twelve are thought of as constituting four inflexional triangles. But there is another arrangement of the nine lines remaining after the erasure of one inflexional triangle, which I have not happened to find mentioned, which yet seems the easiest and most natural mode of access to the inflexional triangle itself.

It shall be presupposed known that there are nine inflexional points, and that every line joining two of them contains also a third. Select two inflexional points A, B, and any third C not collinear with the first two. Call these three an *inflexional triplet*. Join them by three lines, and produce BC, CA, AB to meet the cubic in a second inflexional triplet, in the points A_1, B_1, C_1 respectively.*

Repeating the process upon these three, determine a third triplet A_2, B_2, C_2. From these, determine similarly a fourth triplet. Since its points cannot be additional inflexions, nine having been included already; and since they cannot be the points of the second triplet (as is evident from the figure) unless certain inflexions coincide, they must be the

*It is easily seen that A_1, B_1, C_1, and again A_2, B_2, C_2, are not collinear.