Formula (6) shows at once that when n is even (say $n = 2m$), Q_n is exactly divisible by $(1 + x)^{m-1}$. For the numerator has the factor $(1 - x^2)^m$, and the denominator is $(1 + x)(1 - x)^{2m+1}$. Similarly when $n = 2m - 1$, Q_n is divisible by $(1 + x)^{m-1}$; for $D_t^n \frac{1}{1 - \sin t}$ has now the factor $\cos t$ in the numerator, whence Q_n has still in its numerator the factor $(1 - x^2)^m$. For example, from the table

$$Q_5 x = 61 + 150x + 118x^2 + 30x^3 + x^4$$

and this is exactly divisible by $(1 + x)^2$.

HAVERFORD COLLEGE,
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LINE GEOMETRY.

La Géométrie réglée et ses applications. Par G. KOENIGS. Paris, Gauthier-Villars et Fils. 1895. 4to. 146 pp.

The first general impulse to the study of the geometry of the straight line was given by Plücker's posthumous work in 1868–9, which was soon followed by a large number of contributions from German, French and Italian mathematicians ; the English people were soon interested in the application of the new geometry to mechanics and physics.

In the work of Plücker, point and plane coördinates are used nearly throughout the book, and in many cases the notation is so complicated that readers frequently lose interest before the most important parts are reached.

In most of the subsequent contributions the idea of point or plane coördinates is not considered; most of them presuppose a knowledge of the relation between these and the quantities which directly define a straight line, while others define the new coördinates as parameters, without discussing their meaning.

In view of the great interest which the subject has awakened and its fruitful application to the study of curves and surfaces, its analogy to the geometry of the sphere and its assistance to mechanics, one wonders why no elementary and systematic treatise on the subject has appeared. The work of Sturm is the only comprehensive work on line geometry that has as yet appeared, and it is by no means an elementary one. It is purely synthetic in its treatment and presupposes much that is not generally known. The German translation of Salmon, by Professor Fiedler, gives a short introduction, and a more extensive one is found in the second volume of Clebsch-Lindemann.

In 1887 Professor Koenigs commenced a serial paper in the Annales de Toulouse and the book under consideration is the part of that series which appeared in tomes 3, 6 and 7.

The work begins with a very brief historical introduction and the statement that three phases of line geometry are marked by the works of Plücker, Klein and Lie. Since this chapter was written the work of Sturm has appeared, which distinctly represents a new phase. Chapter I. defines the coördinates of a line as the six determinants of the second order formed by the homogeneous coördinates of two points

$$x_1, x_2, x_3, x_4\,;\ y_1, y_2, y_3, y_4,$$

viz :

$$p_{ik} = x_i y_k - x_k y_i \qquad i, k = 1, 2, 3, 4, i \neq k\,;$$

or of two planes

$$u_1, u_2, u_3, u_4\,;\ v_1, v_2, v_3, v_4\,;$$

$$q_{ik} = u_i v_k - u_k v_i.$$

$$P \equiv p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} \equiv Q \equiv q_{12}q_{34} + q_{13}q_{42} + q_{14}q_{23} \equiv 0.$$

$$p_{ik} = \frac{\partial Q}{\partial q_{ik}}\,;\ q_{ik} = \frac{\partial P}{\partial p_{ik}}.$$

The p_{ik}, q_{em} are denoted by r_{ik}, and the quadratic identity by $\omega(r) = 0$; from now on, point and plane coördinates are entirely subordinated.

The analysis of linear transformation of the r_{ik} is next treated, so that all the subsequent discussions are independent of the form of $\omega(r)$; the representation of pencils and hyper-pencils is next obtained, the latter term denoting all the lines in a plane, or all the lines through a point. The chapter closes with the definitions of complex, congruence and series and the geometrical criteria for their degree and class.

Chapter II. begins with an admirably written discussion of poles and polar planes of linear complexes and of conjugate lines; the invariant of the complex is derived and the conjugate of a given line found, with regard to a given linear complex.

Chapter III. is called systems of linear complexes. The first treatment, which is purely analytic, is of the point and plane homography on a straight line. The idea of involution is first introduced in an unhappy way, as the author develops into the principal thought an analogy which Klein had found to exist between linear complexes in involution and metrical geometry of the point.

The usual idea of involution is introduced later, without showing that the two are really identical.

There is a strange intermixture of analytic and geometric methods throughout the chapter; that a congruence has in general two directrices is proven geometrically, while the particular forms are treated algebraically only.

Two features are worthy of remark ; the consistent use of partial differential equations, and the discussion of a system of complexes in involution. At the end of the chapter are added, in tangential coördinates, the equations of a special complex, special congruence, special series and four complexes having two coincident lines in common. Chapter IV. is devoted to infinitesimal geometry; it begins with the series, or a system of lines having but one parameter. The osculating hyperbolæ of a ruled surface are treated at length, and the order of contact which a line may have with the surface is clearly put. The condition that the surface is developable is obtained but not discussed. Here may be pointed out an error on page 61, where the author has confused the direct and the complementary systems of linear complexes which are tangent, at a given line, to the surface. Throughout this chapter the plane of presupposed knowledge is much higher than in any other in the book ; many theorems are merely cited without proof or reference.

It is shown that any line having two parameters has an envelope, which serves to define the focal surface of a congruence. This surface and the surface of singularities are merely proved to exist; no discussion of either being added. Series, expressed by complexes, are not discussed at all.

The congruence is taken up again, defined by two parameters and some differential expressions are derived, the principal one being the condition which must be fulfilled when a congruence has an osculating linear complex.

The next chapter, V., comprises about two-fifths of the entire book. It is devoted to two problems ; linear transformation of the coördinates and the analogy between line geometry and stereographic projection in four dimensions. Only two sets of transformations are considered; those in which the quadratic identity retains the form

$$x_1x_4 + x_2x_5 + x_3x_6 \equiv 0,$$

and those which satisfy the form

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 \equiv 0.$$

The first may be called tetrahedral coördinates; the axes of the six special complexes $x_i = 0$ are the edges of a tetrahedron. The second system, which are named Klein's coördinates, is composed of complexes mutually in involution. By a happily chosen nomenclature these complexes, their congruences and their hyperboloids are beautifully discussed; it is to be regretted that the author makes no application of them to the quadratic complex.

The last subject treated is the transformation which changes the lines of a complex into the points of a sphere in space of four dimensions, then this sphere is stereographically projected upon the ordinary sphere. In this connection the author uses geometrical figures; the other chapters are entirely without them.

One gathers, from various remarks made in the course of the work, that the author had intended to make the treatise much more extensive, especially as the second part of the title is entirely ignored.

Roughly, the book is a reproduction, with some extensions, and some omissions, of parts of three papers by Professor Klein; those in *Math. Ann.*, vol. 2, pp. 203–213, vol. 5, pp. 257–268, and pp. 278–293.

Although the book is the only elementary treatise on the subject, two important fundamental subjects have been entirely omitted, viz., the canonical form of the linear complex,

$$x_1 = kx_2,$$

and the algebraic discussion of the quadratic complex, as given by Klein. Should one use the book to enable him to better understand most of the memoirs on line geometry, it would prove a valuable aid, but read alone, the reader would get but a narrow and one-sided idea of its usefulness.

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CORNELL UNIVERSITY,
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