ON THE STABILITY OF A SLEEPING TOP.

Abstract of a Lecture before the American Mathematical Society at the Princeton Meeting, October 17, 1896.

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IN the four lectures* of the earlier part of the week I have attempted to simplify the formulæ for the motion of a top by turning to account the methods of the modern theory of functions. In treating this problem I have been largely influenced by the consideration that it is desirable on both sides to reinforce the relationships between pure mathematics and mechanics.

To-day I consider from the same standpoint a much more elementary question, which, however, for this very reason serves as a type for many related problems, viz., the stability of a top rotating about an axis directed vertically upward. The point of support we will assume to be fixed. If it were moveable in a horizontal plane, the formulæ would be somewhat more complicated, but the final result would be quite similar to that in the special case.

When the rotation is very rapid the behavior of the top is as if its axis were held fixed by a special force. This idea was employed, for instance, by Foucault (1851); to regard it, however, as an independent mechanical principle, as is done in many presentations of the subject, is, of course, absurd.

The usual mode of attacking the problem is by means of the *method of small oscillations*. If x, y are the horizontal coördinates of the point of support of the top, n its rotational velocity, and P the moment of its weight, then, rejecting higher powers of x and y, we obtain the linear homogeneous differential equations with constant coefficients

$$x'' + ny' - Px = 0,$$
$$y'' - nx' - Py = 0.$$

The terms in x' and y' in these equations are known as the gyroscopic terms. The solutions of the equations involve the characteristic exponent

$$\lambda = \frac{\pm in \pm \sqrt{4P - n^2}}{2}.$$

* Four lectures "On the theory of the top," delivered at the invitation of Princeton University in connection with its sesquicentennial celebration.

With respect to the form of this exponent two cases are customarily distinguished: the *stable* case, $n^2 > 4P$, and the *unstable* case $n^2 \leqq 4P$, the conclusion then being drawn that in the former case actual oscillations take place about the position of equilibrium, while in the latter case the axis moves away indefinitely from the position of equilibrium.

For the stable case we obtain

$$x = a\cos\frac{nt}{2}\cdot\sin\sqrt{\frac{n^2-4P}{4}}\,t,$$

$$y = a\sin\frac{nt}{2}\cdot\sin\sqrt{\frac{n^2-4P}{4}}\,t,$$

where a is a constant of integration.

I will retain the designations "stable" and "unstable" for the cases $n^2 > 4P$ and $n^2 \leqq 4P$, and will then examine whether the motion actually corresponds to the common use of these terms.

From the start this method of small oscillations lies open to severe criticism. In the so-called unstable case it is directly self-contradictory, since the quantities, which in the construction of the differential equation are assumed to be *small*, become after its integration *large*. There is no reason whatever, therefore, for regarding the results as an approximation to the actual conditions. Even in the stable case the method lacks an accurate basis.

Poincaré, in the corresponding questions of astronomy, carries out the development in series to higher terms. But, supposing that these series converge at all, will their region of convergence extend far enough so that the actual character of the motion can be deduced from them? In the case of the top we are relieved of the laborious investigation of this question, inasmuch as the complete integration can be carried out in explicit form.

I propose the following mode of treating the problem. For the sake of simplicity, the moments of inertia of the top about its principal axes are all assumed equal to 1. The axis, being originally vertical, let the polar angles at any time to be ϑ, ψ, and let $\cos\vartheta = u$. The formulæ of integration are then

$$t=\int\frac{du}{\sqrt{U}}, \qquad \psi=n\int\frac{du}{(u+1)\sqrt{U}},$$

where $$U = 2(u-1)\Big(n^2+(Pu-2)(u+1)\Big)$$

The upper end of the axis (apex) of the top describes in all cases on the surface of the circumscribed sphere a rosette consisting of a number of congruent loops. This is still the case when $n=0$, a loop being then identical with a great circle of the sphere. Our interest centres in the question, how *long* these loops are, *i. e.*, to what value $u=e$ does u diminish, beginning with $u=1$. Here $u=e$ is that root of $U=0$ which lies between $u=+1$ and $u=-1$. In order to obtain the width of the loops it would be necessary to discuss the integral ψ.

Introducing v to denote the value when $u=1$ of the angular velocity $d\vartheta\,/\,dt$ of the axis of the top, this being equal to the measure of the lateral impulse by which the axis is carried out of the vertical position, we have from $U=0$, on writing e for u,

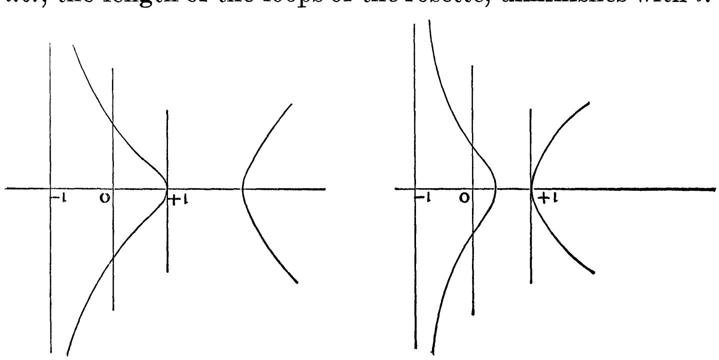
$$v^2 = \frac{(1-e)\,(n^2-2P)\,(e+1)}{e+1}.$$

When e and u are rectangular coördinates, this equation properly interpreted, represents a plane cubic, symmetric to the axis of e, with a vertical tangent at $e=1$, $v=0$, and having $e+1=0$ as an asymptote. This curve has a certain difference of position according as

$$n^2-4P>0 \text{ or } n^2-4P<0,$$

(the case $n^2-4P=0$ may be disregarded for the sake of brevity). In the former (stable) case, the odd branch of the curve passes through $e=+1$, $v=0$, while in the latter (unstable) case, it is the even branch which passes through this point.

In both cases it is the odd branch which is of account for the real motion of the top since $u=\cos\vartheta$ lies, for real ϑ, between -1 and $+1$. In both cases, too, the difference $1-e$, *i.e.*, the length of the loops of the rosette, diminishes with v.



The characteristic distinction between the two cases is this: that for $n^2 - 4P > 0$, the difference $1 - e$ diminishes with v to 0, while if $n^2 - 4P < 0$ this difference never passes a certain lower limit different from 0. Accordingly, in the unstable case, the loops of the rosette take at once a certain finite length even for the smallest lateral impulse given the top.

Theoretically, this furnishes a sharp distinction between the two cases; practically, however, this may become unnoticeable, if $n^2 - 4P$ while < 0, becomes very small in absolute value. The rosette in the unstable case can become as small as we please; and given a stable rosette, a proper choice of the constants n and v will give for the unstable case a rosette *smaller* than the stable one.

Our result is therefore discordant with the common acceptation of the terms "stable" and "unstable." Besides that it does not substantiate the pretensions of the method of small oscillations. If the apex of the top in an unstable case describes a "small" rosette, why does not this fact appear from the method of small oscillations?

The answer to this last question will be apparent, if we introduce the quantity e in the integral t:

$$t = \int \frac{du}{\sqrt{\frac{2(u-e)(1-u)}{1+e}\left(n^2-4P-P(u-1)(e-1)+2(u-1)+2(e-1)\right)}}.$$

The method of small oscillations neglects in the parenthesis

$$n^2 - 4P - P(u-1)(e-1) + 2(u-1) + 2(e-1)$$

the terms containing $u - 1$ and $e - 1$ in comparison with $u^2 - 4P$. This is admissible when and only when $u - 1$ and $e - 1$ being small, $n^2 - 4P$ *is not small*,—and therefore those cases, stable or unstable, where $n^2 - 4P$ is itself a small quantity are incapable of approximate treatment by the method of small oscillations.

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