employed. Attention, however, was called to a fundamental paper of Hölder's in the 34th volume of the *Annalen*. it is important from the Galoisian standpoint: 1, as showing the character of the essential elements of any system of resolvents in which the roots of the given equation can be rationally expressed; 2, as making it imperative to enlarge the notion of a group from a substitution-group whose elements are concrete substitutions on the roots of an equation to a group whose elements are not explicitly given, but merely the laws of their combination.

Professor Pierpont regretted that time did not permit him to develop the theory of finite groups from this abstract standpoint and to touch upon some of the beautiful results obtained by Frobenius, Hölder, Cole and others. The importance of these methods and theories not only for the Galoisian theory, but for many other branches of mathematics, makes it desirable that they be made the subject of a future colloquium.

A GEOMETRICAL METHOD FOR THE TREATMENT OF UNIFORM CONVERGENCE AND CERTAIN DOUBLE LIMITS.

Presented at the Third Summer Meeting of the American Mathematical Society.

BY PROFESSOR W. F. OSGOOD.

The geometrical representation of functions by curves and surfaces is of two-fold importance; for not only does it represent to the eye by means of a concrete picture relations which would otherwise appear only in abstract arithmetic form, but this picture in its turn makes evident new facts and points out at the same time the course that the arithmetic proof of the theorems thus suggested would naturally take. The value of this method for the purposes of instruction alike in elementary and advanced infinitesimal calculus, as well as in analysis generally, can hardly be overestimated. How can the conception of the function be better explained than by such an example as a temperature curve? What better means is there for making clear the idea of the implicit function — y defined implicitly as a function of x by the equation $f(x, y) = 0$ — than by cutting the surface $z = f(x, y)$ by the plane $z = 0$? And how valuable is the surface $u = \varphi(x, y)$ when the differential of a function of two independent variables is introduced!

Nevertheless these geometrical methods have hitherto found but meagre application in the study of those parts of the calculus and of analysis that recent decades have done so much toward putting on a rigorous foundation. These advances have been published almost universally in arithmetic form and many a casual reader of Dini and Jordan and Stolz has, if the writer mistakes not, carried away the idea that this part of analysis consists in proving by means of ε's and inequalities theorems that are self-evident,* or that can be proved satisfactorily by the old-fashioned methods.† This is due partly to the fact that theorems (like the first of those just mentioned) belonging didactically to an advanced stage of analysis, are brought in, for the sake of a logically systematic development, at the beginning; partly to the author's neglect to make clear the necessity of proof, or the fallacy in the proofs usually given. So far as simple (as distinguished from multiple) limits are concerned, ε-proofs are seldom necessary; for the ordinary rules for working with limits — such as, for example, that if α, β each converges toward a limit, then $\lim \alpha\beta = (\lim \alpha) \cdot (\lim \beta)$ — almost always suffice. Moreover if the calculus is taught with a view to giving the student a thorough command of its conceptions in their bearing on the problems of physics and geometry to which it owes its existence, the range of the principles treated in an introductory course can hardly extend beyond simple limits.

In problems involving multiple limits, however, the ε-method becomes valuable, because it allows a complex problem to be reduced to a series of simple ones. But even in the case of some important problems in double limits, graphical methods enable us to grasp the problem as a whole and indicate at the same time a chain of geometrical reasoning, each link of which is capable of immediate translation into arithmetic form. It is the object of this paper to explain the use of these methods in the study of certain well-known problems of analysis that have long since been solved.‡ In the second paper announced for this meeting

* *e. g.* the theorem that a continuous function reaches its maximum and its minimum, and takes on every intermediate value.

† *e. g.*
$$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial^2 u}{\partial y\, \partial x}$$

‡ The methods described and the diagrams contained in this paper have for several years been used for the purposes of instruction in the courses given at this University which treat the subjects to which they apply.

the value of these methods as a means of research is shown by the new results to which they have there led.

Two early papers may be cited which were important contributions to the arithmetic treatment of some of the subjects with which this paper deals and to the study of which the geometric methods here set forth can in turn be advantageously applied. They are: Hankel's *Untersuchungen über die unendlich oft unstetigen und oscillirenden Functionen*, Tübingen, 1870; and Darboux's *Mémoire sur les fonctions discontinues*, Ann. d. l' École Norm. sup. 2ᵉ Ser., vol. IV, pp. 57–112; the paper is dated January 1874. The second of these papers stands in closer relation to the subjects treated in the present paper.

Four Problems of the Calculus.

1. When we pass beyond the rudiments of the infinitesimal calculus, four important problems meet us, namely to ascertain when it is allowable

(1) to integrate a series term by term ;

(2) to differentiate a series term by term;

(3) to reverse the order of integration in a double integral;

(4) to differentiate under the sign of integration.

These are all examples of Double Limits and it is important to recognize them as such. If $f(x, y)$ is a function of the two independent variables x, y, then

$$\lim_{y=0} [\lim_{x=0} f(x, y)] \qquad \text{and} \qquad \lim_{x=0} [\lim_{y=0} f(x, y)]$$

may or may not be the same thing.

Example. $$f(x, y) = \frac{x-y}{x+y}$$

Here the first double limit has the value -1, the second, the value $+1$.

It is to the study of these problems and that of uniform convergence, which plays an important rôle in their treatment, that the geometrical methods above referred to are to be applied.

The Integration of Series Term by Term.

2. Let the series

$$u_1(x) + u_2(x) + \cdots\cdots$$

converge toward a limit, $f(x)$, when the real variable x lies in the interval (a, b):

$$f(x) = u_1(x) + u_2(x) + \cdots\cdots, \qquad a \leqq x \leqq b,$$

and let $f(x)$, $u_i(x)$ be continuous functions of x throughout this interval. Let

$$s_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x)$$

$s_n(x)$ is also continuous. Then

$$\int_a^b f(x)dx \quad \text{means} \quad \int_a^b [\lim_{n=\infty} s_n(x)]\,dx,$$

while
$$\int_a^b u_1(x)\,dx + \int_a^b u_2(x)\,dx + \cdots$$

means
$$\lim_{n=\infty}\left[\int_a^b s_n(x)dx\right]$$

And since the integral is the limit of a sum, we have here to do with a double limit. *The question of whether a series can be integrated term by term is the question of whether it is allowable to reverse the order in a certain double limit — the question of whether the equation*

$$\int_a^b \left[\lim_{n=\infty} s_n(x)\right] dx = \lim_{n=\infty}\left[\int_a^b s_n(x)dx\right]$$

is true.

Example. Let* $\quad s_n(x) = nxe^{-nx^2},$

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* It may seem at first sight as if this series were artificial, since

$$u_1(x) + u_2(x) + \cdots + u_n(x) = s_1(x) + [s_2(x) - s_1(x)] + \cdots + [s_n(x) - s_{n-1}(x)] \equiv s_n(x)$$

the terms cancelling so that the series shuts up like a telescope. But in fact this is only a question of the *form* in which the terms are written, and *any series whatsoever* can be written in the form of a telescope series by this very formula. Thus the series

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$$

when written in the form

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots$$

becomes a telescope series in which $s_n = 1 - \frac{1}{n+1}$

In the case before us the term $s_n(x) - s_{n-1}(x)$ might be written as a definite integral or a Fourier's series and then this *same* series would not appear as a telescope series.

$$u_1(x)=s_1(x), \quad u_n(x)=s_n(x)-s_{n-1}(x), \qquad n>1$$

Then
$$f(x)=\lim_{n=\infty} s_n(x)=0$$

for all values of x and hence

$$\int_0^1 [\lim_{n=\infty} s_n(x)]\, dx = 0$$

On the other hand

$$\int_0^1 s_n(x)\, dx = \tfrac{1}{2}(1-e^{-n})$$

and
$$\lim_{n=\infty}\left[\int_0^1 s_n(x)\, dx\right] = \tfrac{1}{2}$$

Thus this series, although a series of continuous functions converging absolutely toward a continuous limit, cannot be integrated term by term.

The Curve $y=f(x)$ and the Approximation Curves $y=s_n(x)$.

3. Since for every value of x $s_n(x)$ converges, as n increases, toward $f(x)$ and since both $s_n(x)$ and $f(x)$ are continuous functions, it is natural to think of the curve $y=s_n(x)$ as

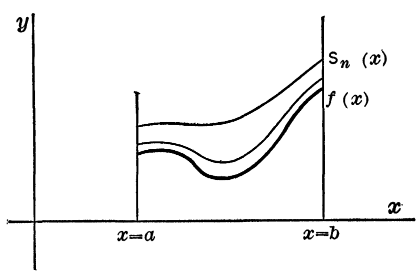

Fig. 1.

approaching in shape ever more and more nearly the limiting curve $y=f(x)$, the properties of $s_n(x)$ going, so to speak, continuously over into those of $f(x)$. "Whatever is true of the variable is true of the limit" is a principle that is applied pretty freely in mathematical physics. But nothing could be further from the truth, as I propose to show by the aid of some very simple figures.

It is possible to throw the question of § 2 into simple geometrical form. Since

$\int_a^b [\lim_{n=\infty} s_n(x)]\, dx$ is the area under the limiting curve

$y = f(x)$ in the interval (a, b),

$\int_a^b [s_n(x)]\, dx$, on the other hand, the area under the approximation curve $y = s_n(x)$, the question reduces to the following: *When will the area under the approximation curve $y = s_n(x)$ approach as its limit the area under the limiting curve $y = f(x)$?*

If the approximation curves were represented in character by Fig. 1, this would always be the case. But the Example of § 2 shows that this cannot be so. Let us see what the reason is.

The Curve $y = nxe^{-nx^2}$. Begin by putting $n = 1$: $y = xe^{-x^2}$. This curve is indicated as the curve (1) of Fig. 2. To pass to the general curve (2): $y = nxe^{-nx^2}$ it is sufficient to divide the abscissas and multiply the ordinates of (1) by $\sqrt{n}$.

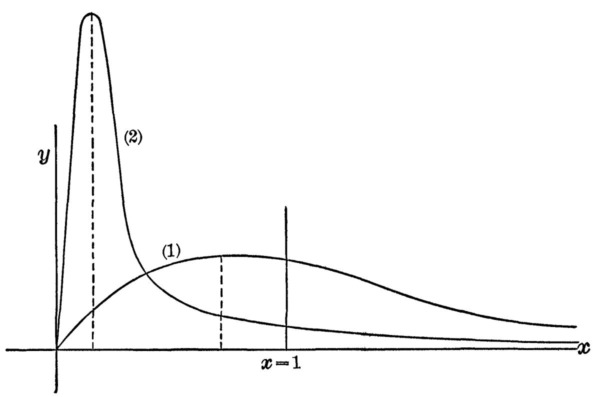

Fig. 2.

It is now easy to explain the contradiction that this example presents to the principle above referred to. The peaks rise higher and higher as n increases, but if $x_0 > 0$ be chosen ever so small and n is then only taken large enough, the peaks will lie to the left of x_0 and $s_n(x_0)$ will approach 0 as its limit. But the *area* under $s_n(x)$ does not approach 0.

Example. Plot the approximation curves for the series whose $s_n(x) = \dfrac{n^2 x}{1 + n^3 x^2}$ and show that this series can be integrated term by term.

Discontinuous Functions defined by Convergent Series of Continuous Functions.

4. Let the series of continuous functions $u_i(x)$:

$$u_1(x) + u_2(x) + \cdots\cdots$$

converge for all values of x in the interval (a, b) and let the limit be represented by $f(x)$. Then, according to the principle that "whatever is true of the variable is true of the limit," $f(x)$, being the limit toward which a continuous function $s_n(x)$ converges, must itself be a continuous function. Let us see whether this is so.

Example. $$s_n(x) = x^{\frac{1}{2n-1}}$$

When $x > 0$, $$x^{\frac{1}{2n-1}} = e^{\frac{1}{2n-1}\log x}$$

and $$\lim_{n=\infty} s_n(x) = 1.$$

When $x = 0$, $$s_n(x) = 0, \qquad \lim_{n=\infty} s_n(x) = 0.$$

When $x < 0$, $$x^{\frac{1}{2n-1}} = -(-x)^{\frac{1}{2n-1}} = -e^{\frac{1}{2n-1}\log(-x)}$$

and $$\lim_{n=\infty} s_n(x) = -1.$$

Thus
$$\begin{aligned} f(x) &= 1, &&\text{when } x > 0;\\ &= 0, &&\text{“} \quad x = 0;\\ &= -1, &&\text{“} \quad x < 0; \end{aligned}$$

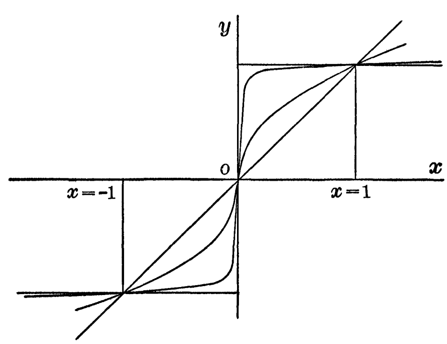

FIG. 3.

and the above principle is again found to be false. The approximation curves $y = s_n(x)$ and the limiting curve $y =$

$f(x)$ are indicated in Fig. 3, a glance at which explains the whole matter.

Example 1. $s_n(x) = (\sin \pi x)^{\frac{1}{2n-1}}$.

Plot the approximation curves and the limiting curve.

Example 2. $u_1(x) + u_2(x) + \cdots =$

$$(1 - x) + (x^2 - x^3) + (x^4 - x^5) + \cdots, \qquad 0 \leqq x \leqq 1.$$

Uniform Convergence.

5. In each of the above examples the series

$$u_1(x) + u_2(x) + \cdots$$

has been a convergent series of continuous functions. If a degree of accuracy for the convergence, chosen at pleasure, is to be attained; I mean, if the positive quantity ε is chosen arbitrarily and the remainder of the series after the first m terms is to be numerically less than ε, then *for any assigned value of* x, $x = x_0$, m can of course be so determined as to satisfy this requirement. This is but a restatement of the condition that the series converges for every value of x from $x = a$ to $x = b$. But for different values of x_0, m will in general have different values. *That which is characteristic of the convergence in each of the examples above studied is that,* ε *being chosen at pleasure, no value of* m *can be found that will fit all values of* x_0 *at once.*

Reference to the diagrams makes this fact immediately evident. For draw the curves $y = f(x) + \varepsilon$, $y = f(x) - \varepsilon$. Then it is clear that m cannot be taken so large that the approximation curve $y = s_n(x)$ will lie wholly within the belt thus marked off. If the approximation curve is to lie within such a belt, no matter how small ε was taken, a further condition than merely that of convergence is necessary, and this condition is afforded by the requirement of *uniform* convergence.

Definition of Uniform Convergence. The series

$$u_1(x) + u_2(x) + \cdots$$

(or the function $s_n(x)$) is said to be uniformly convergent in the interval (a, b) if, the positive quantity ε having been chosen at pleasure, it is then possible to choose m so that

$$| \, s_{m+p}(x) - s_m(x) \, | < \varepsilon, \qquad p = 1, 2, \cdots,$$

no matter what value x may have in the interval (a, b).

That which is essential in this definition is the *order* of the choice of ε, m, x, namely: *first* ε, *secondly* m, the inequality then holding for (*thirdly*) *any* x.

If p be allowed to increase indefinitely, then, since $\lim_{p=\infty} s_{m+p}(x) = f(x)$, from the above inequality it follows that

$$|f(x) - s_m(x)| \leqq \varepsilon.$$

From the combination of these two relations it appears that

$$|f(x) - s_{m+p}(x)| < 2\varepsilon, \qquad p = 1, 2, \cdots$$

or *the remainder after* n *terms*, $r_n(x) = f(x) - s_n(x)$, *is numerically less than* 2ε *if* $n \geqq m$:

$$|r_n(x)| < 2\varepsilon, \qquad n \geqq m.$$

On some accounts it is preferable to make this property the basis of the definition of uniform convergence.

Let us now consider the effect of requiring of the u-series, in addition to the continuity of its terms, that it shall be uniformly convergent. Choose ε at pleasure, determine m so that

$$|s_{m+p}(x) - s_m(x)| < \varepsilon, \qquad p = 1, 2, \cdots$$

for all values of x in the interval (a, b), and plot the curve $y = s_m(x)$. If now a belt is marked off above and below this curve, bounded by the curves $y = s_m(x) + \varepsilon$, $y = s_m(x) - \varepsilon$, then all subsequent approximation curves $y = s_{m+p}(x)$ will lie within this belt and the limiting locus $y = f(x)$ itself will lie within or at most on the boundary of the belt. (Fig. 4.)

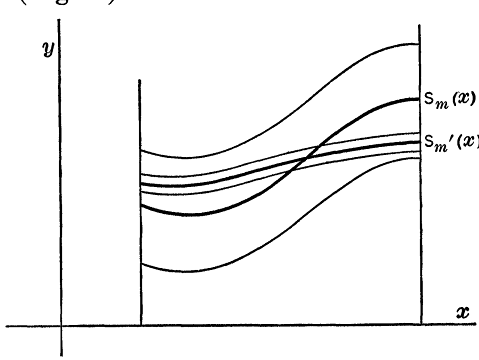

FIG. 4.

Next choose a smaller value $\varepsilon' < \varepsilon$ and a corresponding m' (which will in general be greater than m) and repeat the above construction. The higher approximation curves $y = s_{m'+p}(x)$ and the locus $y = f(x)$ thus lie in the narrower strip.

Let this step be repeated again and again. Thus a set of strips is obtained, each narrower than its predecessor and containing all the higher approximation curves as well as the limiting locus $y = f(x)$. The geometric picture thus brought before the eye shows clearly that the limiting locus itself is a continuous curve. But more than this: *this picture suggests the form of the arithmetic proof.* Arithmetically it is necessary to show that, the positive quantity η having been chosen at pleasure, δ can then be so determined that

$$|f(x_0 + h) - f(x_0)| < \epsilon, \qquad |h| < \delta.$$

This means geometrically that if a strip is marked off

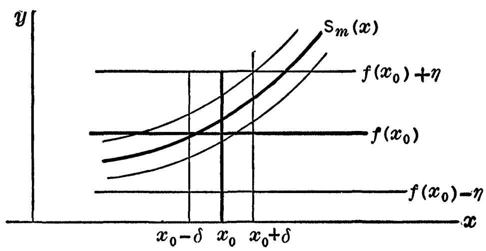

FIG. 5.

bounded by the parallels $y = f(x_0) + \eta$, $y = f(x_0) - \eta$, it will then be possible to determine an interval $(x_0 - \delta, x_0 + \delta)$, such that the points of the locus $y = f(x)$ will, for all values of x within this interval, lie within this strip. Evidently then it is only necessary to take $\epsilon < \frac{1}{2}\eta$; for then the boundaries of the belt about $y = s_m(x)$ will each cut the line $x = x_0$ *within* the strip in question, the distance of such a point of intersection from the nearer boundary of the strip being at least $\eta - 2\epsilon$; and hence they must remain within this strip for all values of x in the interval $(x_0 - \delta, x_0 + \delta)$, if δ is so taken that

$$|s_m(x_0+h) - s_m(x_0)| < \eta - 2\epsilon, \qquad |h| < \delta \tag{1}$$

But
$$|f(x) - s_m(x)| \leqq \epsilon$$

for all values of x; in particular

$$|f(x_0+h) - s_m(x_0+h)| \leqq \epsilon \tag{2}$$

$$|f(x_0) - s_m(x_0)| \leqq \epsilon \tag{3}$$

The combination of (1), (2), (3) gives

$$|f(x_0+h)-f(x_0)| < \eta, \qquad |h| < \delta, \qquad q. e. d.$$

Let the result just obtained be stated in the following:

THEOREM: *A uniformly convergent series of continuous functions is itself a continuous function.*

From this follows at once that *if a convergent series of continuous functions is discontinuous, it must converge non-uniformly.*

The above theorem is virtually a theorem stating the equality of two double limits. It says that if $s_m(x)$ converges uniformly toward $f(x)$, then $\lim_{x=x_0} f(x) = f(x_0)$, and this equation can be written in the form:

$$\lim_{x=x_0} [\lim_{n=\infty} s_n(x)] = \lim_{n=\infty} [\lim_{x=x_0} s_n(x)]$$

Example of a Series defining a Continuous Function and Non-Uniformly Convergent in Every Interval.

6. The examples hitherto considered in the text have been such that only in intervals containing in their interior or at an extremity a certain point x_0 ($x_0 = 0$ in each example) is the convergence non-uniform. I will now give an example of a function $s_n(x)$ non-uniformly convergent in *every* interval.

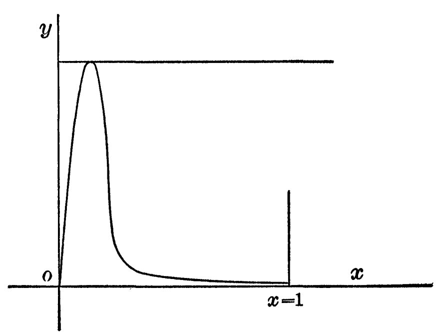

FIG. 6.

Consider the function

$$\sqrt{2e}\, n\, x\, e^{-n^2 x^2}$$

It converges toward 0 for every value of x. The approximation curves are shown in Fig. 6. In this function replace x by $\sin^2 \pi x$ and let

$$\varphi_n(x) = \sqrt{2e}\, n \sin^2 \pi x \cdot e^{-n^2 \sin^4 \pi x}$$

Then the approximation curves for the interval (0, 1), to which we may confine ourselves, since the function has 1 as its period, are indicated in Fig. 7. Next consider the curves

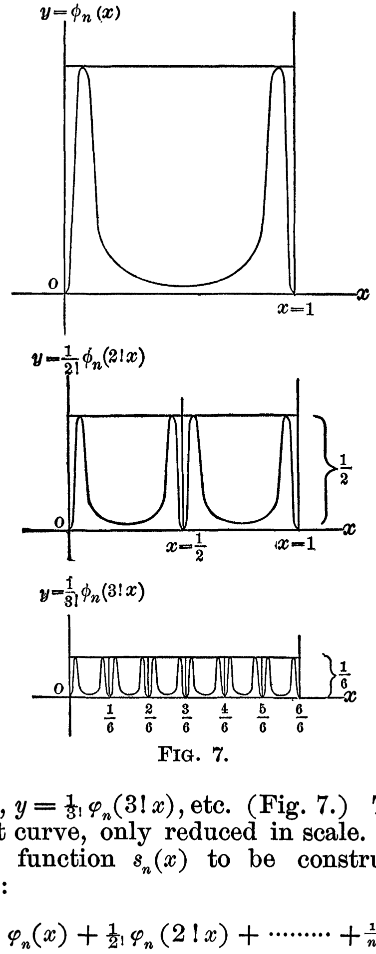

FIG. 7.

$y = \frac{1}{2!}\varphi_n(2!\,x)$, $y = \frac{1}{3!}\varphi_n(3!\,x)$, etc. (Fig. 7.) They are similar to the first curve, only reduced in scale. Out of these functions the function $s_n(x)$ to be constructed is built up as follows :

$$s_n(x) = \varphi_n(x) + \tfrac{1}{2!}\varphi_n(2\,!\,x) + \cdots\cdots\cdots + \tfrac{1}{n!}\varphi_n(n\,!\,x)$$

This function $s_n(x)$ converges toward the limit 0 for every value of x, when n becomes infinite, but the convergence is non-uniform in every interval. It is easy to see how the approximation curves look. The location and height of the higher peaks is determined essentially by the earlier terms of the sum that defines $s_n(x)$; for, all the terms that come after the k^{th} put together cannot equal the quantity

$$\frac{1}{(k+1)!}+\frac{1}{(k+2)!}+\cdots\cdots\cdots\cdots,$$

no matter what n may be, and this quantity is less than $1/k\cdot k!$ From this it follows that, if the positive quantity ε be chosen at pleasure, only a finite number of peaks will rise above the line $y=\varepsilon$. For let k be so taken that $1/k\cdot k! < \frac{1}{2}\varepsilon$ and then held fast. If n is large, the curve

$$y=\varphi_n(x)+\tfrac{1}{2!}\varphi_n(2!\,x)+\cdots+\tfrac{1}{k!}\varphi_n(k!\,x)$$

will rise above the line $y=\frac{1}{2}\varepsilon$ at most in the neighborhood of each of the points

$$0,\ \frac{1}{k},\ \frac{2}{k},\ \cdots\cdots\ \frac{k}{k}=1,$$

and it is then at most in these neighborhoods that $s_n(x)$ can rise above $y=\varepsilon$. (E. g., let $\varepsilon=\frac{1}{8}$, $k=3$. Fig. 8 indicates

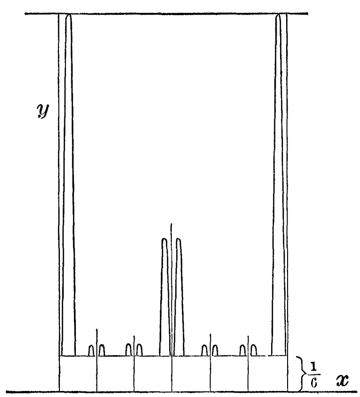

FIG. 8.

that part of $s_n(x)$, $n\geqq 3$, that lies above $y=\varepsilon$.) Moreover the extent of the base of each of these peaks that stands on the line $y=\varepsilon$ contracts and at the same time moves toward its corresponding point $x_0=j/k$, when n increases. Thus it appears that if x' is any value of x, m can be so chosen that $s_n(x')<\varepsilon$, if $n>m$; *i. e.*, $\lim\limits_{n=\infty} s_n(x')=0$; or, dropping the accent, $\lim\limits_{n=\infty} s_n(x)=f(x)=0$. $s_n(x)$ converges, therefore, toward a continuous limit. But the convergence is non-uniform in every interval (a, b), as inspection of the approximation curves shows. For if $x_0 = p/q$ is any rational value of x within this interval, p, q being integers prime to each other, and k is the smallest integer for which $k\,!$ is divisible by q, then the term $\frac{1}{k!}\,\varphi_n(k\,!\,x)$ will give rise to peaks of altitude $\frac{1}{k!}$, the other terms adding to the heights of these peaks, and $s_n(x)$ will have peaks in the neighborhood of x_0. The convergence is then non-uniform in this interval.*

It is to be noticed that the upper limit for the maximum heights of the peaks is different in different intervals. The problem of the most general manner of the convergence of a continuous function $s_n(x)$ toward a continuous limit $f(x)$ is studied in the writer's second paper above referred to.

The proof just given is essentially geometric, in that every step was suggested by direct inspection of the figures that appear on the paper. But the step once given by intuition was capable each time of immediate translation into arithmetic form, no geometric process being used that had not its precise counterpart in arithmetic, and for this reason the proof is as rigorous as if it had actually been thrown into arithmetic form.

Example 1. Let

$$\varphi_n(x) = \sqrt{2e} \cdot n \sin 2\pi x \cdot e^{-n^2 \sin^2 2\pi x},$$

$s_n(x)$ being defined as above. Study the approximation curves $y = s_n(x)$ and show that $s_n(x)$ converges in every interval non-uniformly toward 0.

Example 2. Let

$$s_n(x) = \cos^n \pi x + \tfrac{1}{2!} \cos^n 2\,!\,\pi x + \cdots + \tfrac{1}{n!} \cos^n n\,!\,\pi x$$

Plot the approximation curves and hence show that $s_n(x)$ converges toward a limit for every value of x and that the limit is discontinuous for every rational value of x, but continuous for every irrational value. Indicate by a figure the limiting locus.

Test for Uniform Convergence.

7. A sufficient condition for the uniform convergence of the series

$$u_1(x) + u_2(x) + \cdots$$

* The method by which this example was constructed is virtually that known as Hankel's *Principle of the Condensation of Singularities*. Cf. Hankel's memoir above referred to, or Dini, *Funzioni di variabili reali*, Ch. 9.

is the following. If a set of *constant* positive quantities $M_1, M_2, \cdots$ can be found such that (1)

$$|u_i(x)| \leqq M_i, \qquad a \leqq x \leqq b,$$

and (2) the series $\sum_{i=1}^{\infty} M_i$ converges, then the u-series will converge uniformly. For, ε being given, m can always be so determined that

$$M_{m+1} + M_{m+2} + \cdots + M_{m+p} < \varepsilon, \quad p = 1, 2, \cdots$$

and hence

$$|u_{m+1}(x) + u_{m+2}(x) + \cdots + u_{m+p}(x)| < \varepsilon, \quad p = 1, 2, \cdots; \quad a \leqq x \leqq b.$$

This test shows at once that a power series

$$a_0 + a_1 x + a_2 x^2 + \cdots$$

is uniformly convergent within any interval (a, b) included within* its interval of convergence $(-r, r)$: $-r < a < b < r$. For if h is a positive quantity less than r, but greater than the greater of the two quantities $|a|$, $|b|$, then the series

$$a_0 + a_1 h + a_2 h^2 + \cdots$$

converges absolutely and

$$|a_i x^i| < |a_i| \cdot h^i, \qquad a \leqq x \leqq b$$

Hence a power series represents a continuous function within (not necessarily inclusive of the boundary of) its interval of convergence.

It should be noticed that *uniform* convergence has nothing to do with *absolute* convergence. Thus the non-uniformly convergent series discussed in the text of §§ 2, 4 are absolutely convergent for all values of x. On the other hand, from any absolutely convergent series a conditionally convergent series having the same value can be constructed by adding to the terms of the absolutely convergent series respectively the terms of the conditionally convergent series

$$0 = 1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \cdots$$

*An error sometimes made is that of saying that the series *converges uniformly within its interval of convergence* $(-r, r)$. See a review of Forsyth's *Theory of Functions* in this BULLETIN, 2d Ser., vol. I., p. 145.

The condition for uniform convergence will not thereby be affected.

Example. Show that the series

$$\sin x - \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} - \cdots$$

is uniformly convergent.

Uniformly Convergent Series Integrable Term by Term.

8. Inspection of the approximation curves discloses at once that if $s_n(x)$ converges uniformly toward its limit, the area under $s_n(x)$ converges toward the area under $f(x)$. The arithmetic statement of this fact is the

THEOREM: *A uniformly convergent series can be integrated term by term in any finite interval.*

Here again the arithmetic proof is immediately suggested by the geometric picture. For from Fig. 1, it is clear that $\int_a^b s_n(x)\,dx$, $n \geqq m$, will differ from $\int_a^b f(x)\,dx$ by less than the area of the belt of breadth 2ϵ about the curve $y = s_m(x)$; *i. e.*, by less than $2\epsilon\,(b-a)$; hence

$$\lim_{n=\infty} \int_a^b s_n(x)\,dx = \int_a^b f(x)\,dx.$$

The arithmetic form of the proof is then as follows.

$$f(x) = s_n(x) + r_n(x),$$

$$\int_a^b f(x)\,dx = \int_a^b s_n(x)\,dx + \int_a^b r_n(x)\,dx,$$

$$\left| \int_a^b r_n(x)\,dx \right| \leqq \int_a^b | r_n(x) |\,dx < 2\epsilon\,(b-a),$$

since $$| r_n(x) | < 2\epsilon, \qquad n \geqq m.$$

Hence

$$\lim_{n=\infty} \int_a^b r_n(x)\,dx = 0$$

and $$\lim_{n=\infty} \int_a^b s_n(x)\,dx = \int_a^b f(x)\,dx$$

But this theorem does not hold for an integral one of whose limits is infinite, as is shown by the example:

$$\frac{1}{(1+x)^2} + \frac{1}{(2+x)^2} + \frac{1}{(3+x)^2} + \cdots, \qquad 0 \leqq x.$$

This series converges uniformly for all positive values of x. But the term-by-term integral from a positive lower limit a to infinity is a divergent series.

The Differentiation of Series Term by Term.

9. Let the series

$$u_1(x) + u_2(x) + \cdots\cdots \qquad a \leqq x \leqq b$$

converge and let its value be denoted by $f(x)$. Moreover $u_i(x)$ shall have a derivative for all values of x in the interval. Two cases arise and they must be sharply distinguished from each other.

Case A). The function $f(x)$ has a derivative for every value of x in the interval (a, b).

Case B). The function $f(x)$ has, for certain or all values of x in the interval (a, b), no derivative.*

Case A) is of more importance in applied mathematics, for this is the case in which a function whose properties are more or less known and which in particular is known to be continuous and to have a continuous derivative, is developed into a series of simple functions (rational, trigonometric, Bessel's, etc.) We shall confine ourselves to this case and ask: *When will the derivative of the series be given by the series of the derivatives, i. e., when will*

$$f'(x) = u_1'(x) + u_2'(x) + \cdots$$

be a true equation?

$$f'(x) \qquad \text{means} \qquad \frac{d}{dx}\left[\lim_{n=\infty} s_n(x)\right]$$

$$\sum_{i=1}^{\infty} u_i'(x) \qquad \text{means} \qquad \lim_{n=\infty}\left[\frac{d}{dx} s_n(x)\right]$$

and the question is whether

$$\frac{d}{dx}\left[\lim_{n=\infty} s_n(x)\right] = \lim_{n=\infty}\left[\frac{d}{dx} s_n(x)\right]$$

* A simple example of the latter case is the following:

$$s_n(x) = (1 - \cos^n x) \cdot x \sin\frac{1}{x}, \qquad x \neq 0; \qquad s_n(0) = 0.$$

$$f(x) = \lim_{n=\infty} s_n(x) = x \sin\frac{1}{x}, \qquad x \neq 0; \qquad f(0) = 0$$

Thus $f(x)$ has no derivative when $x = 0$, although it is continuous here as elsewhere. $s_n(x)$ always has a continuous derivative.

Since differentiation is a limiting process, we have again before us the question of the equality of two double limits. According to the principle that "whatever is true of the variable is true of the limit," these two limits must be equal. Let us see whether this is so.

Geometrically the left hand double limit means the slope of the limiting curve $y=f(x)$, while the right hand double limit means the limit approached by the slope of the ap-

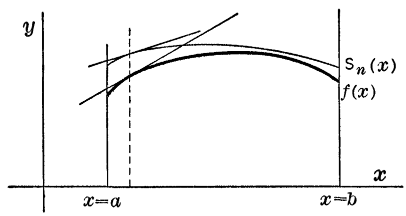

Fig. 9.

proximation curve $y = s_n(x)$. (Fig. 9.)* It is easy to see that these two things are not necessarily the same.

For example, let

$$s_n(x) = n\,x\,e^{-nx^2}$$

(Fig 2.) Then

$$\frac{d}{dx} s_n(x)\,|_{x=0} = n\,(1-2\,nx)\,e^{-nx^2}\,|_{x=0} = n$$

and the slope of the approximation curves at the origin increases without limit. Thus the corresponding series of the derivatives is divergent in this case. But $f(x) = 0$ and hence $f'(x) = 0$.

It would be a mistake however to suppose that the non-uniformity of the convergence is to blame for this result. A glance at the approximation curves in the case that

$$s_n(x) = n \sin^2\pi x\, e^{-n\sin^4\pi x}$$

(Fig. 7) shows that, although the convergence here is non-uniform, their slope approaches 0 for every value of x, and the corresponding series of the derivatives converges toward the right value, 0.

On the other hand, the u-series may converge uniformly

* Professor Byerly has for many years made use of such figures as this in his lectures to explain why some series cannot be differentiated term by term. Cf. his treatise on Fourier's Series and Spherical Harmonics, where these figures appear.

and still not be capable of being differentiated term by term. For example, begin with the curve

$$y = n x^3$$

When n increases, this curve approaches the y-axis. Turn the curve through an angle of, say, $-\frac{1}{4}\pi$ about the origin:

$$x + y = \tfrac{1}{2} n (x - y)^3$$

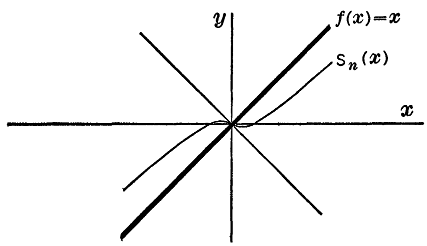

FIG. 10.

(Fig. 10) and let the y of this equation be taken as $s_n(x)$. Then

$$\lim_{n=\infty} s_n(x) = f(x) = x$$

and $f'(x) = 1$. But

$$\frac{d}{dx} s_n(x) \mid_{x=0} = -1$$

and hence at the origin the corresponding series of the derivatives converges, but does not converge toward the right value. Nevertheless, the series converges uniformly.

A simpler example, so far as the analytic formulas are concerned, is the following:

$$s_n(x) = xe^{-n^2x^2}, \quad f(x) = 0; \quad \frac{d}{dx} s_n(x) \mid_{x=0} = 1.$$

Hence $\quad f'(0) = 0, \quad$ but $\quad \lim_{n=\infty} \frac{d}{dx} s_n(x) \mid_{x=0} = 1$

The approximation curves $y = xe^{-n^2x^2}$ are all similar to the curve $y = xe^{-x^2}$, only reduced in scale. The convergence is uniform.

The conclusion to be drawn from these examples is that the uniform convergence of a series is neither a necessary nor a sufficient condition that it may be differentiated term by term.*

Example. Show that the series whose

$$s_n(x) = \cos\pi x \sin^3\pi x \cdot n\, e^{-n \sin^4 \pi x}$$

can be differentiated, but not integrated term by term.

A Sufficient Condition for the Differentiation of a Series Term by Term.

10. Such a condition is given by the following theorem. This theorem also finds application in Case B).

THEOREM. *Let the series*

$$u_1(x) + u_2(x) + \cdots$$

converge for at least one value of x, x_0, *in the interval* (a, b) *and let* $u_i(x)$ *be continuous and have a continuous derivative* $u_i'(x)$ *in this interval; finally let the series of the derivatives*

$$u_1'(x) + u_2'(x) + \cdots\cdots$$

converge uniformly. Then the u-series converges for all values of x *in the interval* (a, b); *and if its value be denoted by* $f(x)$, $f(x)$ *has a continuous derivative, given by differentiating the u-series term by term:*

$$f'(x) = u_1'(x) + u_2'(x) + \cdots$$

For let the value of the u'-series be denoted by $\varphi(x)$:

$$\varphi(x) = u_1'(x) + u_2'(x) + \cdots$$

Then $\varphi(x)$ by § 5 is a continuous function of x and by § 8

$$\int_{x_0}^{x} \varphi(x)\, dx = \int_{x_0}^{x} u_1'(x)\, dx + \int_{x_0}^{x} u_2'(x)\, dx + \cdots$$

$$= [u_1(x) - u_1(x_0)] + [u_2(x) - u_2(x_0)] + \cdots$$

The series $u_1(x_0) + u_2(x_0) + \cdots$ is convergent, hence $u_1(x) + u_2(x) + \cdots$ is convergent, and if its value is denoted by $f(x)$ then

$$\int_{x_0}^{x} \varphi(x)\, dx = f(x) - f(x_0)$$

* In this paper we are dealing only with real functions of a real variable and such functions need not be analytic. If a series of functions, each *analytic* throughout the same *two-dimensional* region of the complex plane, converges uniformly throughout this region, then it is a well-known theorem that the series can be differentiated term by term at all points within this region.

The left hand side of this equation is a continuous function having as its derivative the continuous function $\varphi(x)$. Hence $f(x)$ is continuous and

$$f'(x) = \varphi(x) \qquad q. e. d.$$

This theorem affords a simple proof that a power series can be differentiated term by term.

It will be seen that this theorem is not particularly well adapted to Case A); it assumes more than is given in requiring the series of the derivatives to converge uniformly and does not make use of all that is given, namely that the u-series converges and that its value is a continuous function having a continuous derivative. In the writer's second paper above referred to a theorem is established that conforms more closely to the data in Case A). It may be stated in brief as follows: *If $f(x)$ is a continuous function of x having a continuous derivative and if $f(x)$ is developed into a series of continuous functions having continuous derivatives:*

$$f(x) = u_1(x) + u_2(x) + \cdots\cdots;$$

if furthermore the result of differentiating this series term by term is a convergent series whose value $\varphi(x)$ is a continuous function of x, then the given series can be differentiated term by term:

$$f'(x) = \varphi(x) = u_1'(x) + u_2'(x) + \cdots$$

Thus all that is demanded here beyond what is already given is the continuity of $\varphi(x)$; and I can show by an example that without this requirement (or its equivalent) the theorem would not be true in any interval whatever.

It is worthy of note that in the proofs of both of these theorems, the *integral*, defined as the limit of a sum, was fundamental, the *derivative* appearing as the inverse of the integral.

In most of the examples used in this paper $\lim_{n=\infty} s_n(x) = f(x) = 0$ and it may seem as if this were a very special case. In fact, however, this is not so; for if $f(x)$ is not 0, then we may introduce the new function $S_n(x) = s_n(x) - f(x)$; $\lim_{n=\infty} S_n(x) = 0$, and that which was essential in the manner of the convergence of $s_n(x)$ toward its limit will be preserved, so far as the questions here considered are concerned, in the manner of the convergence of $S_n(x)$ toward its limit, 0. And conversely, each of the examples here considered, where $f(x) = 0$, can be converted into an example where $f(x)$ is not 0 by simply adding to $s_n(x)$ an arbitrarily chosen function $\varphi(x)$:

$$\bar{s}_n(x) = s_n(x) + \varphi(x), \quad \lim_{n=\infty} \bar{s}_n(x) = \varphi(x) \neq 0.$$

Reversal of the Order of Integration in a Double Integral.

11. An important class of double integrals is the following:

$$\int_{x_0}^{x} dx \int_0^{\infty} f(x, y)\, dy,$$

where $f(x, y)$ is a continuous function of the two independent variables x, y throughout the region $a \leqq x \leqq b$, $y \geqq 0$. The question that arises is whether

$$\int_{x_0}^{x} dx \int_0^{\infty} f(x, y)\, dy = \int_0^{\infty} dy \int_{x_0}^{x} f(x, y)\, dx$$

is a true equation, the question of the equality of two double limits. This question can be reduced to that of integrating a series term by term, namely the series

$$\int_0^{\infty} f(x, y)\, dy = \int_0^1 f(x, y)\, dy + \int_1^2 f(x, y)\, dy + \cdots$$

The integral of this series between the limits x_0 and x is the first of the above double limits. On the other hand the term by term integral

$$\int_{x_0}^{x} dx \int_0^1 f(x, y)\, dy + \int_{x_0}^{x} dx \int_1^2 f(x, y)\, dy + \cdots =$$

$$\lim_{n=\infty} \left[\int_{x_0}^{x} dx \int_0^n f(x, y)\, dy \right]$$

has for its value

$$\lim_{n=\infty} \left[\int_0^n dy \int_{x_0}^{x} f(x, y)\, dx \right] = \int_0^{\infty} dy \int_{x_0}^{x} f(x, y)\, dx$$

the interchange of the order of integration in the case of the integral in brackets being here allowable because the integrand is continuous and the limits of integration *finite*. But this last expression is the second of the above double limits, and thus the main question has been reduced to that of integrating the series

$$\int_0^\infty f(x, y)\, dy = \int_0^1 f(x, y)\, dy + \int_1^2 f(x, y)\, dy + \cdots$$

term by term.

Stolz has communicated an example*, due to du Bois Reymond, in which the reversal of the order of integration is not allowable. In the notation of this paragraph it would be as follows:

$$f(x, y) = \frac{\partial}{\partial y}\, \frac{3x^2y^3}{1 + x^6y^6}$$

Then
$$\int_{x_0}^{x} dx \int_0^\infty f(x, y)\, dy = 0;$$

but
$$\int_0^\infty dy \int_{x_0}^{x} f(x, y)\, dx = \pi, \qquad \text{if } x_0 < 0 < x.$$

If $x_0 = 0$,
$$\int_0^\infty dy \int_0^x f(x, y)\, dx = \frac{\pi}{2},\ 0,\ -\frac{\pi}{2},$$

respectively, according as $x > 0$, $= 0$, or < 0.

Conversely, the problem of integrating a series of continuous functions:

$$f(x) = u_1(x) + u_2(x) + \cdots\cdots\cdots$$

term by term can be reduced to the problem of reversing the order of integration in the double integral:

$$\int_{x_0}^{x} dx \int_0^\infty f(x, y)\, dy,$$

where $f(x, y)$ is a continuous function of the two independent variables x, y throughout the region $a \leqq x \leqq b$, $y \geqq 0$.

For let $\bar{f}(x, y) = u_i(x), \qquad a \leqq x \leqq b, \qquad i - 1 \leqq y < i$

and cut the surface

$$z = \bar{f}(x, y)$$

by the plane $x = x_0$. The intersection is indicated in Fig. 11. Then

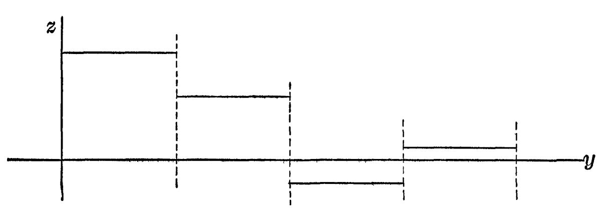

FIG. 11.

* *Fortschritte d. Math.*, Vol. 7, p. 157.

$$u_1(x_0) + u_2(x_0) + \cdots + u_n(x_0) = \int_0^n \bar{f}(x_0, y)\, dy$$

and
$$f(x_0) = \int_0^\infty \bar{f}(x_0, y)\, dy$$

The surface $z = \bar{f}(x, y)$ is discontinuous. But it is easy to replace it by a continuous surface $z = f(x, y)$ such that the relation

$$f(x) = \int_0^\infty f(x, y)\, dy$$

will still hold. For consider the part of the intersection of $z = \bar{f}(x, y)$ with $x = x_0$ lying in the interval from $y = i - 1$ to $y = i$. Let it be replaced by a curve (Fig. 12) which (1) goes

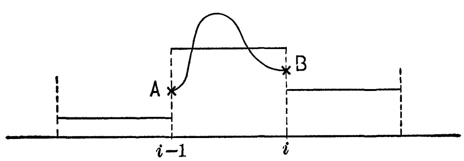

FIG. 12.

through the points A, B (whose coördinates are respectively $y = i - 1$, $z = \frac{1}{2}[u_{i-1}(x_0) + u_i(x_0)]$ and $y = i$, $z = \frac{1}{2}[u_i(x_0) + u_{i+1}(x_0)]$); (2) has its tangents at these points parallel to the y-axis; and (3) includes the same area as the line $z = u_i(x_0)$, *i. e.*, if $z = \psi(y)$ is the equation of this curve,

$$\int_{i-1}^{i} \psi(y)\, dy = \int_{i-1}^{i} u_i(x_0)\, dy = u_i(x_0)$$

In particular, let the curve used for this purpose be

$$z = \alpha y^4 + \beta y^3 + \gamma y^2 + \delta y + \varepsilon.$$

The conditions are just sufficient to determine the coefficients $\alpha, \cdots \varepsilon$ completely.

This replacement being made in each interval: $i = 1, 2, \cdots$, let the curve thus obtained define the function $f(x_0, y)$. Then, dropping the subscript from x_0, we have

$$u_i(x) = \int_{i-1}^{i} f(x, y)\, dy,$$

$$f(x) = \sum_{i=1}^{\infty} u_i(x) = \int_0^\infty f(x, y)\, dy,$$

as is readily proved, and it remains only to show that the function $f(x, y)$ thus defined is a continuous function of the two independent variables x, y. Geometrically this fact is obvious and the arithmetic proof is not difficult.

The result here obtained may be stated as follows: *If*

$$u_1(x) + u_2(x) + \cdots$$

is any series of functions continuous throughout the interval (a, b): $a \leqq x \leqq b$ *; then there exists a function* $f(x, y)$ *of the two independent variables* x, y, *continuous throughout the region* $a \leqq x \leqq b, y \geqq 0$, *and such that*

$$u_i(x) = \int_{i-1}^{i} f(x, y)\, dy.$$

Moreover, if the u-series converges toward a limit $f(x)$,

$$f(x) = \int_0^{\infty} f(x, y)\, dy.$$

Returning now to the main question, namely whether

$$\int_{x_0}^{x} f(x)\, dx = \int_{x_0}^{x} u_1(x)\, dx + \int_{x_0}^{x} u_2(x)\, dx + \cdots$$

we see that the expression on the right hand side can be written in the form

$$\lim_{n=\infty} \left[\int_{x_0}^{x} dx \int_0^n f(x, y)\, dy \right] = \lim_{n=\infty} \left[\int_0^n dy \int_{x_0}^{x} f(x, y) dx \right]$$
$$= \int_0^{\infty} dy \int_{x_0}^{x} f(x, y)\, dx$$

and hence the question is reduced to that of determining whether

$$\int_{x_0}^{x} dx \int_0^{\infty} f(x, y) = \int_0^{\infty} dy \int_{x_0}^{x} f(x, y)\, dx$$

is a true equation.

Differentiation under the Sign of Integration.

12. An important class of improper * integrals for which the question of differentiation under the sign of integration arises is the following :

$$\int_0^{\infty} f(x, y)\, dy,$$

* Riemann's distinction between *eigentliche* and *uneigentliche* integrals marked a step in advance in the integral calculus. In connection with this paragraph see Stolz, *Diff.- u. Int.-Rechnung*, vol. I, ch. X.

where $f(x, y)$, $\dfrac{\partial f(x, y)}{\partial x}$ are continuous functions of the two independent variables x, y throughout the region $a \leqq x \leqq b$, $y \geqq 0$. The question is whether

$$\frac{d}{dx}\int_0^\infty f(x, y)\, dy = \int_0^\infty \frac{\partial f(x, y)}{\partial x}\, dy$$

is a true equation,—again the question of the equality of two double limits,—and it can be reduced to the question of differentiating a series term by term, namely the series

$$\int_0^\infty f(x, y)\, dy = \int_0^1 f(x, y)\, dy + \int_1^2 f(x, y)\, dy + \cdots$$

Here, as in § 9, two cases corresponding to Cases A), B) arise, and we will restrict ourselves to the case that $\int_0^\infty f(x, y)\, dy$ has a derivative. Then this derivative,

$$\frac{d}{dx}\int_0^\infty f(x, y)\, dy,$$

is the first of the above double limits. On the other hand, the term by term derivative of the series:

$$\frac{d}{dx}\int_0^1 f(x, y)\, dy + \frac{d}{dx}\int_1^2 f(x, y)\, dy + \cdots =$$

$$\lim_{n=\infty}\left[\frac{d}{dx}\int_0^n f(x, y)\, dy\right]$$

has for its value

$$\lim_{n=\infty}\left[\int_0^n \frac{\partial f(x, y)}{\partial x}\, dy\right] = \int_0^\infty \frac{\partial f(x, y)}{\partial x}\, dy,$$

differentiation under the sign of integration being here allowable because $\dfrac{\partial f(x, y)}{\partial x}$ is a continuous function of the two independent variables x, y, and the limits of integration are both *finite*. But this last expression is the second of the above double limits, and thus the main question has been reduced to that of differentiating the series

$$\int_0^\infty f(x, y)\, dy = \int_0^1 f(x, y)\, dy + \int_1^2 f(x, y)\, dy + \cdots$$

term by term.

A simple example of a case in which differentiation under the sign of integration is not allowable is the following. It is readily shown, either by evaluating the indefinite integral or by a simply device, that

$$\int_0^\infty x^3 e^{-x^2 y}\, dy = x$$

for all values of x. Hence

$$\frac{d}{dx}\int_0^\infty x^3 e^{-x^2 y}\, dy = 1$$

But
$$\int_0^\infty \frac{\partial}{\partial x}(x^3 e^{-x^2 y})\, dy = \int_0^\infty (3x^2 - 2x^4 y) e^{-x^2 y}\, dy$$

and when $x = 0$, the value of this expression is 0.

The writer has given a sufficient condition for differentiating an improper integral under the sign of integration.*

Conversely, the problem of differentiating a series term by term can, at least in the case that the derivatives of the terms are continuous functions, be reduced to that of differentiating

$$\int_0^\infty f(x, y)\, dy$$

under the sign of integration, $f(x, y)$, $\frac{\partial f(x, y)}{\partial x}$ being continuous functions of the two independent variables x, y. For by § 11

$$u_i(x) = \int_{i-1}^{i} f(x, y)\, dy,$$

where $f(x, y)$ is continuous, and if $u_i'(x)$, $i = 1, 2, \cdots$ is a continuous function of x, then $\frac{\partial f(x, y)}{\partial x}$ is a continuous function of x, y regarded as independent variables, and hence

$$u_i'(x) = \int_{i-1}^{i} \frac{\partial f(x, y)}{\partial x}\, dy$$

* Cf. Stolz, vol. II., p. 333; or *Monatshefte f. Math. u. Phys.* 7, p. 90; 1896.

The main question is whether, if $f(x)$ has a derivative,

$$f'(x) = u_1'(x) + u_2'(x) + \cdots$$

is a true equation. The right-hand side can be written in the form

$$\lim_{n=\infty} \left[\frac{d}{dx}\int_0^n f(x,y)\, dy\right] = \lim_{n=\infty}\left[\int_0^n \frac{\partial f(x,y)}{\partial x}\, dy\right]$$
$$= \int_0^\infty \frac{\partial f(x,y)}{\partial x}\, dy,$$

and thus the question is reduced to that of whether

$$\frac{d}{dx}\int_0^\infty f(x, y)\, dy = \int_0^\infty \frac{\partial f(x,y)}{\partial x}\, dy$$

is a true equation.

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LINEAR DIFFERENTIAL EQUATIONS.

Einleitung in die Theorie der linearen Differentialgleichungen mit einer unabhängigen Variablen. Von DR. LOTHAR HEFFTER. Leipzig, Teubner, 1894. 8vo, XIV+258 pp.

IN teaching higher mathematics, the question presents itself, to what functions beyond the algebraic and elementary transcendental functions should the student be introduced first? The answer which is given to this question almost as a matter of course is: the elliptic and then the Abelian functions. Without in any way casting doubt upon the wisdom of the choice here expressed for many cases (perhaps even for most cases as far as the elliptic functions go), it may be pointed out that the above is by no means the only satisfactory answer, and that the explanation of its almost universal acceptance is to be found in great part in mere tradition. Another class of functions which forms from many points of view an equally satisfactory introduction to the study of the higher transcendental functions, is the class with which the book under review deals, *i. e.*, functions defined by homogeneous linear differential equations. Not only is this true of the study of these functions