It is clear that the number of these groups increases very rapidly with the increase of n. *E. g.*, when $n=1000$, there are $(166+1)(166+2)-2=28,054$ of order 6.

This article may be regarded as a continuation of "*The substitution groups whose order is four*," *Philosophical Magazine*, vol. 41 (1896), pp. 431–437.

PARIS, *May*, 1896.

NOTE ON THE SPECIAL LINEAR HOMOGENEOUS GROUP.

BY PROFESSOR HENRY TABER.

On page 232 of the BULLETIN are given the conditions necessary and sufficient in order that a transformation of the special linear homogeneous group in n variables may be generated by the repetition of an infinitesimal transformation of this group. As a corollary of these conditions it ollows that a transformation of this group can be generated fthus if the multiplicities of the several roots of its characteristic equation have no common factor, or if the roots of its characteristic equation are all equal to $+1$. But one or other of these conditions is satisfied if n is an odd prime. Therefore, if n is an odd prime, every transformation of the special linear homogeneous group in n variables can be generated by the repetition of an infinitesimal transformation of this group, that is, belongs to a continuous one-term sub-group containing the identical transformation.

On the other hand, if $n=2$ or is composite, it follows from the conditions given on page 232 that the special linear homogeneous group in n variables contains an assemblage of transformations no one of which can be generated by the repetition of an infinitesimal transformation of this group. Nevertheless, by the repetition of an infinitesimal transformation of this group we may approximate as nearly as we please to any transformation of this assemblage. Thus corresponding to any transformation A of the special linear homogeneous group that cannot be generated by the repetition of an infinitesimal transformation of this group can always be found a transformation A_ρ of this group, whose coefficients are rational functions of a parameter ρ, such that for all but a finite number of the values of ρ, A_ρ can be generated by the repetition of an infinitesimal transformation of this group, and by taking ρ sufficiently near to some constant value, as zero, A_ρ may be made as nearly as we please equal to A.

The method of demonstrating this theorem in general may be illustrated by one or two examples. Thus, for $n=2$, there is a doubly infinite system of transformations of the special linear homogeneous group no one of which can be generated by the repetition of an infinitesimal transformation of this group. For a proper choice of coördinates any transformation A of this system is defined by the equations

$$x' = -x + y, \qquad y' = -y.$$

Let A_ρ denote the transformation defined by the equations

$$x' = (\rho - 1)x + y, \quad y' = \frac{1}{\rho - 1} y;$$

A_ρ is then a transformation of the group, and passes continuously into the transformation A as ρ approaches zero continuously. For values of ρ sufficiently near to zero, the roots, $\rho-1$ and $(\rho-1)^{-1}$, of the characteristic equation of A_ρ are distinct; and therefore by a change of coördinates A_ρ will be defined by the equations

$$X' = (\rho - 1)X, \qquad Y' = \frac{1}{\rho - 1} Y,$$

where X and Y are the new coördinates of the point represented formerly by the coördinates x, y.*

Let $\rho - 1 = e^{\alpha + \beta\sqrt{-1}}$, where α is real and β is the smallest positive argument of $\rho-1$; and let B_ρ denote the transformation defined by the equations

$$X' = e^{\frac{1}{m}(\alpha + \beta\sqrt{-1})} X, \qquad Y' = e^{-\frac{1}{m}(\alpha + \beta\sqrt{-1})} Y$$

where m is any positive integer. B_ρ is then a transformation of the special linear homogeneous group, and B_ρ^m is equal to A_ρ. By taking m sufficiently great, B_ρ may be made as nearly as we please equal to the identical transformation. Therefore, by the repetition of an infinitesimal transformation of the special linear homogeneous group we may approximate as nearly as we please to A.

If in the m^{th} power of B_ρ we put $\rho=0$, we obtain the finite

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*The formulæ for the transformation of coördinates are

$$x = X - \frac{(\rho-1)}{\rho(\rho-2)} Y, \qquad y = Y.$$

transformation A; but the transformation B_ρ for $\rho=0$ is illusory, since for $\rho=0$ the formulæ for the transformation of coördinates are illusory. This is more readily apparent if we employ the original system of coördinates in the equations determining B_ρ, in which case B_ρ is defined by the equations

$$x'=e^{\frac{1}{m}(\alpha+\beta\sqrt{-1})}x+\frac{2(\rho-1)\sqrt{-1}}{\rho(\rho-2)}\sin\left(\frac{\alpha+\beta\sqrt{-1}}{m}\right)\cdot y,$$
$$y'=e^{-\frac{1}{m}(\alpha+\beta\sqrt{-1})}y.$$

If we regard m as varying continuously, then for any value of ρ other that 0, 1 or 2, the totality of transformations B_ρ constitute a continuous one-term group containing the identical transformation. The transformation A may be said to be a limit of this family of one-term groups, since, for a value of ρ sufficiently near to zero, certain of the transformations of the corresponding one-term group are as nearly equal as we please to A, which, nevertheless, does not belong to any one-term group of this family.*

For $n=6$, among the transformations of the special linear homogeneous group in n variables which cannot be generated by the repetition of an infinitesimal transformation of this group is the transformation A, defined by the equations

$$x_1'=\lambda x_1+x_2,\ x_2'=\lambda x_2+x_3,\ x_3'=\lambda x_3+x_4,\ x_4'=\lambda x_4$$
$$x_5'=\mu x_5+x_6,\ x_6'=\mu x_6,$$

where $\qquad \lambda=e^{\alpha+\beta\sqrt{-1}},\quad \mu=e^{-2\alpha+(\pi-2\beta)\sqrt{-1}}$

(α and β being both real). On the other hand, the transformation A_ρ defined by the equations

$$x_1'=\rho\lambda x_1+x_2,\ x_2'=\frac{\lambda}{\rho}x_2+x_3,\ x_3'=\lambda x_3+x_4,\ x_4'=\lambda x_4,$$
$$x_5'=\mu x_5+x_6,\ x_6'=\mu x_6,$$

which belongs to the special linear homogeneous group, and passes continuously into the transformation A as ρ approaches unity continuously, can, if λ and μ are fixed, for all but a finite number of the values of ρ, be generated by the repetition of an infinitesimal transformation of this group.

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* The transformation

$$x'=x+y,\quad y'=y$$

is also a limit of this family of one-term groups. It is obtained from B_ρ by putting $m=1$, and in the resulting transformation making ρ equal to 2.

Thus the roots of the characteristic equation of A_ρ are $\rho\lambda$, $\rho^{-1}\lambda$ each of multiplicity one and λ, μ each of multiplicity two; therefore, for all values of ρ which render $\rho\lambda$ distinct from the other roots of the characteristic equation; in particular if ρ is sufficiently near to unity, A_ρ belongs to a continuous one-term group containing the identical transformation.* By varying ρ we obtain a family of one-term group, to which, as before, A is a limit.

NOTES.

A REGULAR meeting of the American Mathematical Society was held in New York on Saturday afternoon, May 23, at three o'clock, the President, Dr. G. W. HILL, in the chair. There were thirteen members present. On the recommendation of the Council the following persons, previously nominated, were elected to membership: Mr. LOUIS TRENCHARD MORE, Johns Hopkins University, Baltimore, Md.; Professor HENRY ALLEN PECK, Syracuse University, Syracuse, N. Y. The following paper was read:

Dr. J. E. HILL: "Bibliography of surfaces and of twisted curves."

THE meeting of the British Association for the Advancement of Science will be held this year at Liverpool, opening on September 16. The President is Sir JOSEPH LISTER. Next year the Association will meet in Toronto.

THE German Mathematical Society will meet in conjunction with the German Association of Naturalists and Physicians at Frankfort-on-the-Main, September 21–26.

WE learn from *Nature* that the De Morgan Memorial Medal has this year been awarded by the Council of the London Mathematical Society to Mr. SAMUEL ROBERTS, F. R. S. This medal is awarded every three years. Mr. ROBERTS was one of the earliest members of the Society, and has published numerous papers, many of them of great value. The presentation will be made at the annual meeting of the Society in November next.

COLUMBIA UNIVERSITY. The Department of Mechanics will give the following graduate courses during the year 1896–97: By Professor R. S. WOODWARD: (1) Theory of

* See page 232.