GUNDELFINGER'S CONIC SECTIONS.

Vorlesungen aus der Analytischen Geometrie der Kegelschnitte. Von SIGMUND GUNDELFINGER. Herausgegeben Von FRIEDRICH DINGELDEY. Leipzig, Teubner, 1895. pp. 434.

AMONG recent analytic works on Conic Sections there are at least three for which one is very thankful, — the late Professor Casey's, Miss Scott's, and the one whose title appears above. The plan of this last is to systematically develop the theory by means of homogeneous coordinates, while bringing out the fact that the elementary (x, y) system is merely a case to which we can descend when so minded. This latter may seem a minor point; pedagogically it is not so, and it is certainly not well explained in many books.

The development, then, of the theory is really analytic, though one feels that the analysis is under the control of a masterly geometric insight.

The work divides into two sections and an appendix. The first section begins with the explanation of point-coordinates in all their generality. In spite of the fact that two of the three books above mentioned adopt this plan, it does seem a good method to explain these matters — at all events at first — with a special system of coordinates or "unit-point," and with the "areal" ("barycentric") system for preference, we get the full advantage of a homogeneous system, awkward factors do not appear, and if in any projective question we need an arbitrary unit-point, we have merely to show how to project the fundamental triangle so that the unit-point projects into the centroid. Evidently there is a pedagogic gain, but it is not evident that there is any loss.

We have then the fundamental expression for the distance of a point from a line; and the inference that when the point and line unite, the expression

$$u_1x_1 + u_2x_2 + u_3x_3$$

is zero, whence we have the equation of a line. It is necessary to mention such a detail so long as elementary text-books reverse the order and prove *first* that the equation of a line (p, α) is

$$x \cos \alpha + y \sin \alpha - p = 0,$$

and *second* that the left represents the distance from the line (p, α) to the point (x, y).

The coordinates of a line are next defined as the coefficients u_1, u_2, u_3, and their connection with the distances from the fundamental points explained.

In the next article the angle between two lines is determined and brought into the form

$$\cos^{-1}\frac{\omega(u, v)}{\sqrt{\omega(u, u)}\sqrt{\omega(v, v)}},$$

and the ambiguity arising from the square roots on the one hand and the choice of angles on the other is resolved. The formula for the distance of two points is given much later.

For the curve of the second order $f(x, x) = 0$ (§ 4) it is shown first that it cuts out two points from an arbitrary range $x + y$; next by making the points coincide, the equation of the pair of tangents from y is determined; next comes the double ratios of the two intersections on the one hand, and x, y on the other, and the all-important case $f(x, y) = 0$, when the two pairs are harmonic, is discussed.

From this, by letting y lie on its polar $f(x, y) = 0$, the equation of the tangent is deduced. It is to be observed that the tangent is regarded primarily as a line which meets the curve in coincident points, *not* as a line which unites with its pole.

Soon the discriminant and the line-equation of $f(x, x) = 0$ are introduced, and the condition that two lines u_n, v_n meet on the curve is found in the form

$$\begin{pmatrix} uv \\ uv \end{pmatrix} = 0$$

where the notation expresses the determinant obtained by bordering the discriminant with u_1, u_2, u_3 and v_1, v_2, v_3. When u is given, this is the equation of the two points where u meets the conic, and the resolution of $\begin{pmatrix} uv \\ uv \end{pmatrix}$ into factors is actually effected. This is Gundelfinger's method, and another method (due to Aronhold) is explained whereby also the intersections are expressed in terms of the coordinates of the intersecting line.

In the next article the curve of the second class is similarly handled, without unnecessary repetition of algebra.

Then follows (§ 6) the classification of the conic sections, and this is gone into with fulness.

In § 7 the circle and its points at infinity make their appearance, the former through a line-equation, as the envelope of a line whose distance from a fixed point is constant. We notice that Miss Scott pursues the same plan. The points at infinity then appear of their own accord, as points into whose equations the equation $\omega(u, u) = 0$ breaks up. Then comes the point-equation of a circle of given centre and radius, and thereby of course the distance between two points. Returning to the points at infinity, it is shown that the expression $\omega(u, u)$ is a "definite" form; that is, it cannot change sign for real values of u_1, u_2, u_3. It would perhaps have been better to have established this point by connecting the expression with the squared area of the fundamental triangle. And lastly the angle made by one line with another is connected with a double ratio made by the points at infinity and the circular points.

The following article is devoted to algebra, and contains developments based on Kronecker and Weierstrass of well-known theorems. First a general theorem on "definite" quadratic forms is proved, namely that *if $\phi(u_1, u_2 \cdots u_n)$ is such a form, which vanishes for the values $p_1, p_2 \cdots p_n$ of the variables, of which values at least one is not zero, then the expressions $\phi'(p_1), \phi'(p_2) \cdots \phi'(p_n)$ also all vanish.* This theorem is applied to prove that what Salmon (*Higher Algebra*, lesson vi.) calls the "equation of secular inequalities," namely,

$$\begin{vmatrix} \alpha_{11} - \lambda, & \alpha_{12}, & \cdots \alpha_{1n} \\ \alpha_{21}, & \alpha_{22} - \lambda, & \cdots \alpha_{2n} \\ \vdots & \vdots & \vdots \\ \alpha_{n1}, & \alpha_{n2}, & \cdots \alpha_{nn} - \lambda \end{vmatrix} = 0,$$

where $\alpha_{i\kappa} = \alpha_{\kappa i}$, has only real roots; and further to prove that if the determinant has a factor $(\lambda - \lambda_1)^l$, then the minor of any term has at least the factor $(\lambda - \lambda_1)^{l-1}$ (when $\lambda_1 \lessgtr 0$), and at least the factor λ^{l-2} (when $\lambda_1 = 0$).

In § 9 is an introductory account of invariants, covariants, contravariants, and "mixed concomitants," and it concludes with the reduction of $f(x, x) = 0$ to two terms only, namely, to

$$x_1 x_3 - c x_2^2 = 0,$$

or, as the equation is later written,

$$x_2^2 + 2\, x_1 x_3 = 0,$$

a form which has the advantage of being its own reciprocal. In § 10 the general equation is transformed so as to be referred to the principal axes, with the data necessary for determining the equations and lengths of these axes, etc. Herein the previous algebraic section finds its use, but one cannot avoid regretting the fact that 24 pages are required for a detailed treatment of the transformation.

The next article discusses the forms of the curves from the equations referred to their axes, finds the foci by combining the line-equation with the imaginary circular points, and proves the focus and directrix property, and the fact that the sum or difference of focal radii for the ellipse or hyperbola is constant. The proofs are given always with a view to the general handling of the homogeneous equation; the section is by no means an idle incursion into the field of the elementary text-book.

In § 12 we return to projective problems, and have the generation of a conic by projective pencils or ranges, the remarkable theorem of Pascal and the reciprocal theorem of Brianchon. The developments of Kirkman and others are not gone into.

The second section of the work deals with pencils and nets of conics, and the reciprocal ideas of ranges and webs.

First the three line-pairs of a pencil are found, and the cubic, written

$$\lambda^3 B - 3\lambda^2 \Theta + 3\lambda H - A = 0,$$

which leads to them is carefully discussed. Then it is shown that the double points of these line-pairs form a triad harmonic with all conics of the pencil.

Then, to mention only leading propositions, which will suffice to show the order of ideas, the two conics of a pencil which touch a given line as determined. Thence follows Desargue's theorem that a line is cut in involution by the conics of a pencil. Then it is proved that a line, which cuts from two given conics point-pairs whose double ratios are assigned, envelops a curve of the fourth class, and thence in particular Staudt's conic, the envelope when the point-pairs are harmonic, makes its appearance.

The reciprocal statements as to a range are then made; and special reference is made to the range of confocal conics; *i.e.* the range which includes the circular points. Desargue's theorem, or rather its reciprocal, shows that the two conics through any point are confocal, and Staudt's conic, taken for the circular points and any conic, appears as the director circle of that conic.

In the next place suitable triangles of reference are assigned for the special forms which a pencil assumes owing to coincidences among the ground-points.

Then comes the significance of the vanishing of the intermediate invariants of two conics, the Θ and H of the cubic before mentioned; and the case where one or both of the conics degenerates is not overlooked. Conics whose Θ vanishes are termed conjugate, instead of apolar as with Reye.

The confocal range has now an article to itself, with full treatment of the possible special cases. The important property of the foci, that the product of their distances from any tangent is constant, begins a new article, wherein are discussed the circles having double contact with a given conic.

The next article returns to the pencil of conics, and proves the theorem that the poles of a given line, as to all the conics, lie on a conic N, and discusses especially the case when the given line is the line ∞. The reciprocal proposition that the polars of a given point y, as to a range of conics, envelop a conic N, is applied to a confocal range; the envelope is now called Steiner's parabola. In particular is shown the bearing of this parabola on the problem of the four normals from y to a conic of the range; the feet of the normals are the points of contact with the selected conic of common lines of it and of the Steiner's parabola which arises from the point y.

When the parabola is reciprocated with respect to the conic, it is shown analytically that we have a hyperbola through the centre of the conic and the points at ∞ on its axes. That this rectangular hyperbola — which Casey calls the Apollonian hyperbola — also solves the problem of drawing normals from a point to a conic, is shown by means of an elegant expression for a curve which intersects a curve of degree n at the feet of the normals from a point y. Leaving special considerations, it is then shown that when the conic N is written

$$N_1 u_1 + N_2 u_2 + N_3 u_3 = 0,$$

where u is the given line, the Jacobian of N_1, N_2, N_3 as to x_1, x_2, x_3 is nothing else than the sides of the diagonal triangle of the four ground-points of the pencil. And the article ends with a too brief account of a remarkable combinant conic $\psi=0$, which Casey (*Conics*, second edition, p. 486) calls the fourteen-line conic. As this conic is not perhaps yet well known, it may be worth while to briefly indicate it. Consider the four points a, b, c, d, whose coordinates are λ_1, $\pm\lambda_2$, $\pm\lambda_3$, and the four lines α, β, γ, δ, whose coordinates are $1/\lambda_1$, $\pm 1/\lambda_2$, $\pm 1/\lambda_3$. The fundamental triangle is the common harmonic triangle of the four points and the four lines, and the two systems of four elements have to one another a well-known elementary and perfectly reciprocal projective relation. The conic whose point-equation is

$$x_1^2/\lambda_1^2 + x_2^2/\lambda_2^2 + x_3^2/\lambda_3^2 = 0,$$

and whose line-equation is

$$\lambda_1^2\xi_1^2 + \lambda_2^2\xi_2^2 + \lambda_3^2\xi_3^2 = 0$$

is the conic in question. On each of the four lines are three points, where it is met by the other three lines; the Hessian pair of these points are on the conic. On each side of the fundamental triangle are two point-pairs, such as $\beta\gamma$, $\alpha\delta$, $bd \mid ac$, and $cd \mid ab$ (where $bd \mid ac$ means the intersection of bd and ac). The Jacobian of these two point-pairs, which of course are harmonic pairs, is two points of the conic. Thus the fourteen points are enumerated, and the fourteen lines follow by interchange of point and line. The equations show that when we take four real points or lines the conic is imaginary.

The conic is considered further, in connection with the two "equianharmonic" conics through the four points, on p. 370. The three are written

$$x_1^2 + ex_2^2 + e^2x_3^2 = 0,$$
$$e^3 = 1.$$

Perhaps Wolstenholme's equivalent form,

$$x_1^2 + 2\,x_2x_3 = 0,\ \ x_2^2 + 2\,x_3x_1 = 0,\ \ x_3^2 + 2\,x_1x_2 = 0,^*$$

is more convenient for the treatment of the system. The conic, or rather the combinant which, put zero, gives the conic, affords great service in the discussion (p. 371) of the reality of the intersections of two conics whose point-equations are assigned. The complete solution of this problem is given in a form more simple than that of Kemmer.

Returning to complete our sketch of the order of the book, the radius of curvature, which was mentioned in connection with Steiner's parabola, is found for any algebraic curve whose line-equation is assigned, and the result stated also for a curve of assigned point-equation. The evolute of a conic is also discussed.

The next three articles (§§ 22–25) discuss the Hessian and Cayleyan of a net of conics, and prove (so far as proof is needed) the reciprocal theorems as to the Hessian and Cayleyan of a web. It must be observed that what is here called the Hessian is called by Salmon the Jacobian of the three base-conics (SALMON, *Conics*, p. 360). Especially to be noticed is the method of § 25.

The following will indicate the main result of the article. Consider the net

$$y_1\,(x_1^2 + 2\,mx_2x_3) + y_2\,(x_2^2 + 2\,mx_3x_1) + y_3\,(x_3^2 + 2\,mx_1x_2) = 0,$$

to which form any net may be reduced unless it contain a double line † and the web

$$v_1\,(u_1^2 + 2\,\mu u_2u_3) + v_2\,(u_2^2 + 2\,\mu u_3u_1) + v_3\,(u_3^2 + 2\,\mu u_1u_2) = 0.$$

The condition of conjugacy ($\Theta = 0$) is

$$(y_1v_1 + y_2v_2 + y_3v_3)\,(1 + 2\,m\mu) = 0\,;$$

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* BULLETIN, Series 2, vol. 1, p. 121.

† SALMON, *Conics*, pp. 365, 367. In the equation at top of p. 367, the coefficient m should be inserted.

if, then, $1 + 2m\mu = 0$, each member of the net is conjugate with each member of the web.

On the other hand, considering the cubics

$$x_1^3 + x_2^3 + x_3^3 + 6\,mx_1x_2x_3 = 0,$$
$$u_1^3 + u_2^3 + u_3^3 + 6\,\mu u_1u_2u_3 = 0,$$

whose polar conics and pole conics respectively constitute the net and web, the Hessian of the former is (SALMON, *Curves*, p. 190)

$$-m^2(x_1^3 + x_2^3 + x_3^3) + (1 + 2\,m^3)\,x_1x_2x_3 = 0,$$

and the Cayleyan of the latter (*ib.* p. 191) is

$$\mu\,(x_1^3 + x_2^3 + x_3^3) + (1 - 4\,\mu^3)\,u_1u_2u_3 = 0.$$

These are the same if

$$-1/m^2 - 2/m = 1/\mu - 4/\mu^2,$$

and in particular if

$$1 + 2\,m\mu = 0.$$

Thus if we call the net and web "conjugate" (or apolar) *the Hessian of a net is the Cayleyan of the conjugate web; and reciprocally the Hessian of the web is the Cayleyan of the net.* But the more careful treatment of our book appears necessary in view of the special cases.

The Hessian and Cayleyan are thus the same thing in different aspects; and this idea is fruitfully applied to the Hessian and Cayleyan of a net or of a web.

In the final article (§ 26), the relation of conjugacy is further developed, and applied (for instance) to prove that between the squares of six tangents of a conic there is a linear relation.

Then follows an appendix of 180 pages, consisting for the most part of solved examples, which sufficiently demonstrate the applicability and completeness of the theory previously developed.

A number of problems are taken from Steiner, who, it will be recalled, stated various results without proof. These results of Steiner's are not all, at the present date, very obvious; and the solutions, which are partly due to Dr. Gundelfinger, partly to Dr. Dingeldey, are both direct and elegant. It is, I think, fair to say that until the appearance of the books named at the beginning, students were well able to handle conics with homogeneous point-coordinates, and to reciprocate the theorems arrived at; but the use of point and line coordinates combined was not fairly presented. For this reason (not to mention others) these solutions will repay the attention of students.

A point of historical interest in the Geometry of the Triangle should be mentioned. It appears that Steiner came across the "cosine circle" of a triangle (p. 316), that is, he noticed as a special case of one of his theorems that there is only one point p (the point now called the symmedian) through which if lines be drawn so that the intercepts made on them by the pairs of sides of a triangle are bisected at p, the ends of the intercepts lie on a circle, whose centre is of course p.

Finally we have the handling of two integrals, the second of which

$$\int \frac{\Sigma \pm (c_1 x_2 dx_3)}{[\frac{1}{2} c_1 g'(x_1) + \frac{1}{2} c_2 g'(x_2) + \frac{1}{2} c_3 g'(x_3)] \sqrt{f(x, x)}}$$

(where $f(x, x)$ and $g(x, x)$ are ternary quadratic forms, x_1, x_2, x_3, are the coordinates of a point on $g(x, x) = 0$, c_1, c_2, c_3, the coordinates of a fixed point on the same conic), is transformed into

$$-\int \frac{d\lambda}{\sqrt{g(\lambda)}}$$

where $g(\lambda)$ is the cubic which determines the line-pairs of the pencil defined by f and g. This covers, for example, the problem of transforming the elliptic integral $\int (xdx)/\sqrt{a_x^4}$ to Weierstrass's normal form.

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ON DIVERGENT SERIES.

BY PROFESSOR A. S. CHESSIN.

THAT every semi-convergent series can by a proper arrangement of its terms be made divergent is a well-known fact. It will be shown in this note that, conversely, every divergent series which does not tend towards infinity (series oscillating between finite limits) can by a proper arrangement of its terms be made convergent.

Only series with real terms will be considered since the investigation of series with complex terms, at least with regard to the substance of this note, can be reduced to that of series with only real terms.

THEOREM I. — *An infinity of numbers being given within a limited interval, we know that there will be at least one infinite accumulation of numbers of the given totality within the given interval. Let, in general, $N_1, N_2, \cdots, N_g$ be the numbers about which these infinite accumulations take place. It is always possible to form g distinct convergent series having for their respective sums the numbers $N_1, N_2, N_3, \cdots, N_g$.*