ON THE INTRODUCTION OF THE NOTION OF HYPERBOLIC FUNCTIONS.*

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THE difficulties in the way of a satisfactory geometrical deduction of the fundamental formulæ of the hyperbolic functions seem to be due to the lack of a definition of these functions which shall be independent of the particular position of the argument area. A general definition of this kind can, however, readily be found in terms of the ratios of certain *areas*, instead of *lines*. From this definition the addition-theorem and other characteristics can be easily deduced by the methods of analytic geometry; and the definitions hold, furthermore, not merely for the rectangular, but for *any* hyperbola.

I. *The circular functions.* In order to bring out clearly the analogy with the circular functions, I will first indicate briefly how the latter would be defined according to this method.

In a circle of radius a (Fig. 1) let ϕ be the angle between the radii OP and OQ, and let OP' be drawn perpendicular to

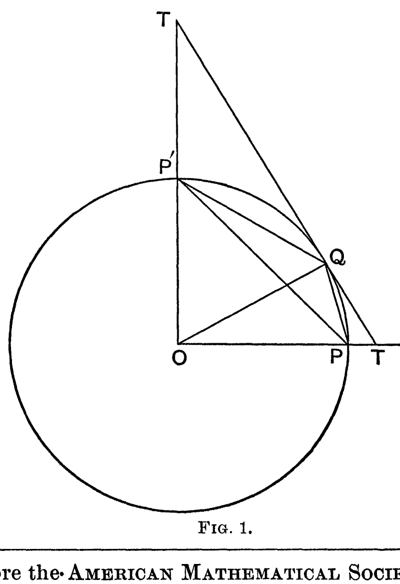

FIG. 1.

* Read before the AMERICAN MATHEMATICAL SOCIETY, December 28, 1894. For various geometrical definitions of these functions, see Professor A. Macfarlane's paper: "On the definition of the trigonometric functions," 1894.—EDITORS.

OP. The following areas are either well known or easily found (the sign $\triangle$ denoting triangle):

$$\text{sector } OPQ = \tfrac{1}{2}a^2\phi, \qquad \triangle\, OPQ = \tfrac{1}{2}a^2 \sin\phi,$$

$$\triangle\, OQP' = \tfrac{1}{2}a^2\cos\phi, \quad \triangle\, OPP' = \tfrac{1}{2}a^2,$$

from which follow immediately:

$$\phi = \frac{\text{sector } OPQ}{\triangle\, OPP'}, \quad \sin\phi = \frac{\triangle\, OPQ}{\triangle\, OPP'}, \quad \cos\phi = \frac{\triangle\, OQP'}{\triangle\, OPP'}.$$

Similarly, if the tangent at Q meet OP in T and OP' in T', it is not difficult to show that

$$\tan\phi = \frac{\triangle\, OTQ}{\triangle\, OPP'}, \qquad \text{ctn}\,\phi = \frac{\triangle\, OQT'}{\triangle\, OPP'},$$

$$\sec\phi = \frac{\triangle\, OTP'}{\triangle\, OPP'}, \qquad \csc\phi = \frac{\triangle\, OPT'}{\triangle\, OPP'},$$

Now, these formulæ might be taken as *definitions* of the argument ϕ and of its various functions. They can then be immediately extended to any ellipse, the only modification necessary being that OP and OP' shall be conjugate semi-diameters. The area of the triangle OPP' is then $= \tfrac{1}{2}ab$, and ϕ is equal to the difference between the eccentric angles P and Q.

II. *Definition of the hyperbolic functions.* Since of two conjugate diameters only one meets the hyperbola in real points, the conjugate hyperbola must be employed also, and P' is the point where the diameter conjugate to OP meets the conjugate hyperbola. We shall then define the argument u and its functions in strict analogy with the preceding results, as follows (see Fig. 2):

$$u = \frac{\text{sector } OPQ}{\triangle\, OPP'}, \quad \sinh u = \frac{\triangle\, OPQ}{\triangle\, OPP'}, \quad \cosh u = \frac{\triangle\, OQP'}{\triangle\, OPP'},$$

etc.

If now the hyperbola be referred to its principal axes as axes of coördinates, its equation may be written

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{1}$$

Let x_1, y_1 be the coördinates of P, and x_2, y_2 those of Q. Then the coördinates of P' will be $\frac{ay_1}{b}, \frac{bx_1}{a}$, and the area of the triangle OPP' is equal to $\frac{1}{2}ab$. The definitions of sinh u and cosh u become

$$\sinh u = \frac{x_1y_2 - x_2y_1}{ab}, \quad \cosh u = \frac{x_1x_2}{a^2} - \frac{y_1y_2}{b^2}. \tag{2}$$

Interchanging the coördinates of P and Q, we have

$$\sinh(-u) = -\sinh u, \quad \cosh(-u) = \cosh u. \tag{3}$$

Also, if P and Q coincide, $u = 0$ and

$$\sinh 0 = 0, \quad \cosh 0 = 1. \tag{4}$$

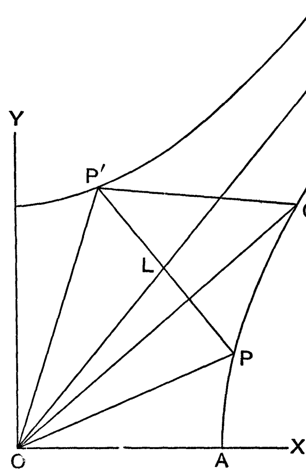

FIG. 2.

Let $u_1 = \frac{\text{sector } OAP}{\triangle OPP'}$ and $u_2 = \frac{\text{sector } OAQ}{\triangle OPP'}$, so that $u = u_2 - u_1$. The coördinates of A being a, 0, we have

$$\sinh u_1 = \frac{y_1}{b}, \ \cosh u_1 = \frac{x_1}{a}; \ \sinh u_2 = \frac{y_2}{b}, \cosh u_2 = \frac{x_2}{a}. \tag{5}$$

Comparing with (1), we see at a glance that

$$\cosh^2 u_1 - \sinh^2 u_1 = 1, \tag{6}$$

and that this result is general may be instantly verified by (2), for

$$\cosh^2 u - \sinh^2 u = \frac{(b^2x_1x_2 - a^2y_1y_2)^2 - a^2b^2(x_1y_2 - x_2y_1)^2}{a^4b^4}$$

$$= \frac{b^2x_1^2 - a^2y_1^2}{a^2b^2} \cdot \frac{b^2x_2^2 - a^2y_2^2}{a^2b^2} = 1.$$

III. *The Addition-Theorem.* Substituting (5) in the definitions of (2), we have

$$\left.\begin{aligned} \sinh(u_2 - u_1) &= \sinh u_2 \cosh u_1 - \cosh u_2 \sinh u_1 \\ \cosh(u_2 - u_1) &= \cosh u_2 \cosh u_1 - \sinh u_2 \sinh u_1 \end{aligned}\right\} \quad (7)$$

Writing $+ u_1$ instead of $- u_1$, we have by (3)

$$\left.\begin{aligned} \sinh(u_2 + u_1) &= \sinh u_2 \cosh u_1 + \cosh u_2 \sinh u_1 \\ \cosh(u_2 + u_1) &= \cosh u_2 \cosh u_1 + \sinh u_2 \sinh u_1 \end{aligned}\right\} \quad (8)$$

The generality of these formulæ is easily verified in the manner just exemplified in formula (6).

IV. It is clear from (5) that the definitions we have given reduce to the ordinary form for the special position there considered. It remains to be shown that in the form given $\sinh u$ and $\cosh u$ are really functions of u alone. To this end let us choose the asymptotes as axes of coördinates. Let α and β be the coördinates of any point, and let ω be the angle between the asymptotes; the equation of the hyperbola is then

$$\alpha\beta = \tfrac{1}{2}ab \csc \omega. \quad (9)$$

The coördinates of P being $\alpha_1 = OL$ and $\beta_1 = LP$, those of Q being α_2 and β_2, those of P' will be α_1 and $-\beta_1$, and we have

$$\sinh u = \frac{\alpha_2\beta_1 - \alpha_1\beta_2}{ab \csc \omega}, \quad \cosh u = \frac{\alpha_2\beta_1 + \alpha_1\beta_2}{ab \csc \omega}, \quad (10)$$

or, if we write $\dfrac{\alpha_2}{\alpha_1} = \lambda$ and therefore $\dfrac{\beta_2}{\beta_1} = \lambda^{-1}$,

$$\sinh u = \tfrac{1}{2}(\lambda - \lambda^{-1}), \quad \cosh u = \tfrac{1}{2}(\lambda + \lambda^{-1}). \quad (11)$$

If we now apply to the plane the linear transformation

$$\alpha' = k\alpha, \quad \beta' = k^{-1}\beta, \tag{12}$$

the hyperbola is transformed into itself. The points P and Q are moved to any arbitrary position on the curve, but the ratio $\alpha_2 : \alpha_1$ is unaltered. It is easily shown that in this transformation the area of any triangle, and hence the area of any figure in the plane, is unchanged; so that u, sinh u and cosh u are unchanged. They are therefore all functions of the ratio $\alpha_2 : \alpha_1$ alone. Hence sinh u and cosh u are functions of u alone, and the definitions here given are a proper generalization of the usual definitions.

V. *The exponential formulæ.* The sector OPQ may be regarded as the limit of a circumscribed polygon and hence u may be regarded as the limit of the sum of a series of hyperbolic sines. To make each of the terms in this series equal, we have evidently only to put $\lambda = \rho^n$, where n may be any whole number. Then, writing *lt* briefly for *limit,*

$$u = \tfrac{1}{2}\,\mathrm{lt}\; n(\rho - \rho^{-1}) = \mathrm{lt}\,\frac{n}{2}(\lambda^{\frac{1}{n}} - \lambda^{-\frac{1}{n}}) \tag{13}$$

$$= \mathrm{lt}\,\frac{\lambda^{\frac{2}{n}} - 1}{\frac{2}{n}}\,.\,\lambda^{-\frac{1}{n}}, \text{ as } n \text{ increases indefinitely.}$$

The limit of $\lambda^{-\frac{1}{n}}$ is equal to 1, and the limit of $\dfrac{\lambda^{\frac{2}{n}} - 1}{\frac{2}{n}}$ is the natural logarithm of λ. Hence $\lambda = e^u$. Introducing this value of λ in (12), we have

$$\sinh u = \tfrac{1}{2}(e^u - e^{-u}), \quad \cosh u = \tfrac{1}{2}(e^u + e^{-u}). \tag{14}$$

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