

## JORDAN'S COURS D'ANALYSE.

*Cours d'Analyse de l'École Polytechnique.* Deuxième édition, entièrement refondue. By C. JORDAN. Vol. I. Paris, Gauthier-Villars, 1893. 8vo. vi + 612 pp.

M. JORDAN'S *Cours d'Analyse* ranked as a classic from the time of its publication; and deservedly so, for it had all the excellencies of style and treatment that we have come to expect in the publications of the leading French mathematicians. The first two volumes were, in essence, a reproduction of lectures that had been given at the *École Normale*, and were intended to serve as an introduction to the infinitesimal calculus and to the theory of functions of a complex variable. To avoid daunting his readers M. Jordan was compelled to pass somewhat lightly over the discussion of first principles, but as far as was compatible with elementary instruction his explanations of fundamental theorems were logically exact. In the first volume of the new edition of his *Cours d'Analyse*, M. Jordan has introduced much new matter, and has made many changes in arrangement; for instance, he now discusses the simpler functions of the complex variable and the branches of an algebraic function as early as the second chapter, and he has incorporated in the main text the greater portion of the long note at the end of the third volume of the first edition. But more important than any alterations in detail, however extensive they may be, is the fact that the book in its present shape appeals to a new class of readers. It is intended for those that are familiar with the ordinary processes of the differential calculus, but unfamiliar with modern researches on the underlying ground-principles. Professor Klein has said very justly\*: "The second edition of the *Cours d'Analyse* of Camille Jordan may be regarded as an example of extreme refinement in laying the foundations of the infinitesimal calculus. To place a work of this character in the hands of a beginner must necessarily have the effect that at the beginning a large part of the subject will remain unintelligible, and that, at a later stage, the student will not have gained the power of making use of the principles in the simple cases occurring in the applied sciences," but he adds, "on the other hand it is a matter of course that for more

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\* The Evanston Colloquium; Lectures on mathematics delivered at Northwestern University by Felix Klein, and reported by A. Ziwet, page 49. New York, Macmillan, 1894.

advanced students, especially for professional mathematicians, the study of works like Jordan's is indispensable."

The present volume contains five chapters which treat of real variables, complex variables, series, applications of Taylor's series, and plane algebraic curves.

The earlier pages of the first chapter are concerned with irrational numbers, limits, assemblages, restricted functions, continuous functions, and functions with restricted variation. When the corresponding ideas have been fully explained, the author proceeds to establish many of the usual theorems of the differential calculus on the lines marked out in the first fifty pages. As we have said above, M. Jordan's object is to explain the first principles of the subject with all possible rigor and generality. To effect his purpose he begins with the definition of the irrational number and thus paves the way for the discussion of limits, assemblages, functions, and integration regarded as a summation. Metaphysical difficulties arise when an analysis based on number-concepts is bound up with a geometry based on concepts of continuous magnitude; but it is possible to evade these difficulties. Wherever an appeal is ordinarily made to geometric experience, recourse may be had to a definition; a familiar example is that of the irrational number. The number  $\sqrt{2}$  was unquestionably suggested by geometric experience, but it is possible to frame such a definition of irrational numbers in general as will permit  $\sqrt{2}$  to find its place as a formal number in the number-system. When sufficient definitions have been granted as a foundation it becomes possible to build a superstructure that is logically unassailable. Such is the plan followed by many writers on real functions of real variables. This avoidance of direct reference to experience gives the subject a purely formal aspect, and the value of a subject so treated must be gauged largely by the correspondence of its definitions with the known facts of consciousness. But the moment such a test is applied we are again face to face with the metaphysical difficulties. This in nowise detracts from the value of such works as the one that we are now considering. It is of high importance that we should be able to say in the Calculus, as in Euclidean and non-Euclidean geometry, that from certain initial data all other theorems and consequences can be deduced by processes of pure reasoning.

The theory of assemblages is an essential element in the working out of M. Jordan's scheme. An assemblage  $E$  is said to be perfect when it *contains* its first derived assemblage  $E'$ , whereas Cantor regards  $E$  as perfect only when it *coincides with*  $E'$ . According to M. Jordan's definition a perfect assemblage may contain isolated points. Suppose, for simplicity, that  $E$  is of one dimension and that the points of  $E$

are strewn along a finite segment  $AB$  of a straight line; then the remaining points of the segment constitute a complementary assemblage  $E_1$ . Points belonging at once to  $E$ ,  $E_1'$ , or to  $E_1$ ,  $E'$ , where  $E'$ ,  $E_1'$  are the first derived assemblages from  $E$ ,  $E_1$ , are called frontier points of  $E$ . All other points are said to be interior or exterior to  $E$ . It is possible that  $E$  may have no interior or exterior points, but it must have frontier points and these frontier points must form a perfect assemblage. A simple example is offered by the assemblage of rational numbers on  $AB$ ; here all the points of  $AB$  are frontier points. An assemblage of points  $(a, b, c, \dots)$  in any number of dimensions is said to be restricted when the absolute values of the numbers  $a, b, c; \dots$  admit an upper limit  $\mu$ ; and we have the well-known theorem that a restricted assemblage composed of infinitely many points has at least one limiting point. The separation (*écart*) of two points  $(a, b, c, \dots)$  and  $(a', b', c', \dots)$  being understood to mean the value of  $|a - a'| + |b - b'| + |c - c'| + \dots$ , two assemblages  $E, F$  of the same character are said to be separated when the separations of points  $p$  of  $E$  from points  $q$  of  $F$  admit a lower limit  $\Delta$  greater than 0. When the assemblages are restricted and perfect this lower limit is attained; that is, there exist points  $p, q$  with a separation exactly equal to  $\Delta$ .

An assemblage that is perfect and restricted is said to be of a single piece (*d'un seul tenant*) when it cannot be resolved into several separate assemblages. The necessary and sufficient condition that  $E$  shall be of a single piece is that between any two points  $p, p'$  whatsoever of  $E$  it is always possible, whatever be the value of the positive number  $\epsilon$ , to interpolate a chain of intermediate points that belong to  $E$  and are such that the separations of consecutive points are all less than  $\epsilon$ . Such an assemblage  $E$  coincides with  $E'$ ; for were  $E$  to contain isolated points, it would clearly be possible to find a value  $\epsilon$  sufficiently small to break the chain from an isolated point to a second point of  $E$ . We have reproduced this theorem because it serves to illustrate the importance of having definite images before the mind's eye when dealing with abstract propositions of the kind just stated. The use of the word chain might suggest that an assemblage of a single piece is merely an analytic mode of expressing the geometric notion of a connected region, that is, a region in which it is possible to pass from any one point to any other along a continuous curve that lies wholly in the region; but to grasp the full significance of the theorem attention must be paid to the fact that  $E$  is *perfect*. To take a definite example, let all the points  $(a, b)$  whose coördinates are algebraic numbers be excluded from the square whose corners are  $(\pm 1, \pm 1)$ ; it has

been proved by Cantor\* that any two points of the remaining assemblage can be connected by a continuous curve all of whose points belong to  $E$ . This assemblage of points has therefore the property that any one of its points can be chained to any other of its points in the manner described above, but the assemblage is not of a single piece; for each of the excluded points is a limiting point of  $E$ , and therefore  $E$  is not perfect. In the case of one dimension, an assemblage of a single piece that contains  $a$  and  $b$  contains every number between  $a$  and  $b$ . In passing it may be remarked that this result of Cantor's throws light on a theorem to which Professor Klein called attention in his Evanston lectures, viz., that with the exception of the point  $(x = 0, y = 1)$  the exponential curve  $y = e^x$  has no algebraic point.

The interpretation of  $\int f(x)d\sigma$ , where  $d\sigma$  is the element of integration, as the limit of an infinite sum, permitted Riemann to extend the meaning of integration to many cases where  $f$  is discontinuous within the region of integration. This work of Riemann's was generalized and completed by Darboux when he showed that for a restricted function the two sums  $\sum M d\sigma$ ,  $\sum m d\sigma$ , where  $M$  and  $m$  stand for the upper and lower limits of  $f$  in  $d\sigma$ , have always perfectly determinate limits (the integral by excess and the integral by defect). A function may be treated as integrable when these limits coincide. Investigations have been carried out along these lines; of these the following theorem † may serve as a sample. Let the  $n$  integrable functions  $\phi_\kappa(x)$ , where  $\kappa = 1, 2, \dots, n$ , lie between  $\alpha_\kappa$  and  $\beta_\kappa$  inclusive when  $x$  lies between  $a$  and  $b$  inclusive, and let  $F(x_1, x_2, \dots, x_n)$  be continuous for values  $x_\kappa$  between  $\alpha_\kappa$  and  $\beta_\kappa$  inclusive, then  $F(\phi_1 x, \phi_2 x, \dots, \phi_n x)$  is integrable in the interval  $a$  to  $b$ .

So far as concerns the function under the sign of integration these theorems of Riemann and Darboux are satisfactory, but they are not equally clear and precise as regards the field of integration. M. Jordan gives to the field all its generality and shows that it may be regarded as having an interior and also an exterior extent. Taking a region of two dimensions in the finite part of the plane of  $x, y$ , the recognized process for finding the area is to divide the region by parallels to the axes into a network of rectangles; some of these contain only interior points, others contain boundary points. The

\* G. Cantor: "Sur les ensembles infinis et linéaires de points." *Acta Mathematica*, vol. 2, p. 868.

† Du Bois-Reymond: "Ein allgemeiner Satz über die Integrirbarkeit von Functionen integrirbarer Functionen." *Math. Annalen*, vol. 20, p. 122.

limit to which the sum of the former rectangles tends is the area of the region, while the latter sum tends to zero. This analysis covers all cases of common occurrence. Using a different form of language, M. Jordan divides a restricted two-dimensional assemblage of points by means of a network of squares; the points of some of these squares may be all interior to  $E$ , the corresponding squares will then be called interior squares. All the remaining squares are frontier squares. M. Jordan shows that by considering the sums of interior squares for all possible modes of division we must arrive at an upper limit called the interior extent of  $E$ , and that by considering the sums of interior and frontier squares we must similarly arrive at a lower limit called the exterior extent. It is only when the two numbers thus defined coincide that the assemblage  $E$  can be said to have a measurable extent. Although the division is made primarily by means of squares, it is shown afterwards that the same results are attained when the square elements are replaced by other elementary regions that are "quarable." The following is a simple example of a non-measurable assemblage. Let  $E$  consist of points  $(x, y)$  for which  $0 \leq y \leq 1$ ; and let  $x$  go from 0 to 1 inclusive when  $y$  is rational, from 0 to  $-1$  inclusive when  $y$  is irrational. Here the interior extent vanishes, while the exterior extent differs from 0.

Sections VI and VII of chapter I are concerned with derivatives and integrals of functions of a single variable, partial derivatives and total derivatives; section IX treats of continuous lines and rectifiable lines. The discussion of what is meant by the length of a curve is very thorough, and much of the analysis is not to be found in any other text-book. The same remark applies to section XII on change of variables in definite integrals. In this section M. Jordan, to avoid impairing the generality of his work, makes no attempt to reduce the multiple integral to a succession of simple integrals, for this reduction has only been established for integrals, properly so called, taken over measurable fields; in his method all the variables are changed simultaneously.

The second chapter gives an account of complex variables and contains proofs of Cauchy's two cardinal theorems in integration.

The first of these two theorems is stated in the following form: "Let  $C$  be a closed contour which is continuous and has no multiple point, and further let  $C$  be such that all points non-exterior to  $C$  are interior to the domain  $E$ ; then the integral  $\int f(z)dz$  taken over any closed rectifiable line  $K$  interior to  $C$  will be identically zero." Here  $f(z)$  denotes a function of  $z$  which is synectic within the domain  $E$ . Several

pages are devoted to rational functions and algebraic functions; and these are followed by an exposition of the principal properties of the elementary transcendents.

Chapter III opens with proofs of Taylor's theorem for real functions of one and several real variables, and contains proofs of the corresponding theorems for functions of a complex variable. The third section relates to infinite series and products with numerical terms and gives the usual rules and theorems relating to the corresponding kinds of convergence; this leads on to series of functions and to a careful explanation of the method of continuation of integral series. On page 342 we have a statement of Weierstrass's important theorem—"If  $S(u, z_1, z_2, \dots, z_n)$  be an element of an analytic function with  $n + 1$  variables, such that  $S(u, 0, 0, \dots, 0) = 0$  has  $m$  zero roots, then the equation  $S(u, z_1, z_2, \dots, z_n) = 0$  admits, when  $z_1, z_2, \dots, z_n$  are infinitely small,  $m$  infinitely small roots." The proof of this general theorem is not given, but it is used in the special case  $m = 1$  as a basis for the theory of algebraic functions of one variable. When completed by the addition of some auxiliary theorems it permits one to assert with safety that if  $n$  branches of  $u$  derived from the algebraic equation  $S(u, z) = 0$  vanish at  $z = 0$ , then each of these  $n$  branches can be represented in the neighborhood of  $z = 0$  by a convergent series

$$Mz^\mu + Mz^{\mu_1} + \dots,$$

where the  $\mu$ 's are positive increasing fractions such that the least common multiple  $m$  of their denominators is equal to or less than  $n$ . Space is found in this chapter for many special examples of the general theory of series; the ordinary trigonometric expansions are worked out and a short account is given of the hypergeometric series and of the gamma-function. The question of maxima and minima of functions of two variables is examined with care, and the discussion of the doubtful cases leaves few outstanding difficulties. This part of the chapter will repay careful study.

Little need be said of chapter IV, as the text is substantially the same as that of the first edition. As the title of the chapter indicates, this portion of the book indicates the rôle played by the differential calculus in the proof of geometric theorems. The number of illustrations is large and varied, and the proofs are always elegant. An especially interesting part is that which relates to ruled surfaces, congruencies and complexes.

The last chapter, on algebraic plane curves, will be of great assistance in clearing up difficulties that arise in the study of Cremona transformations. It begins with an explanation of homogeneous coördinates and homographic transformations

and introduces the discriminant, Hessian and polars of a ternary form  $f(x, y, z)$ . The second section treats of cycles, and shows the bearing of this method on the problems of finding the numbers of intersections of two curves  $f = 0$ ,  $F = 0$  at a point that is multiple on both, and of finding the reduction in the class of  $f = 0$  produced by the presence of this point. Here an opportunity is presented for defining Halphen's *characteristic exponents*. It is proved (i.) that a point  $p$  of a curve, with its associated cycles  $C_1, C_2, \dots$  of orders  $r_1, r_2, \dots$ , becomes after a homographic transformation a point  $p'$  on the new curve, with the same number of associated cycles  $C'_1, C'_2, \dots$  of the same orders as before; (ii.) that when some of the first cycles have coincident tangents, the same is the case with the transformed cycles; and (iii.) that the characteristic numbers are unaffected. After homographic come quadratic transformations; the effect of such transformations upon cycles variously placed in regard to the triangle of reference is discussed; and a proof is given that it is possible to change any curve by a number of quadratic transformations into another curve whose cycles are of the first order, and whose tangents at the multiple points are distinct. The volume ends with a section on birational transformations of a curve, in which use is made of a particular case of Abel's theorem.

The need for a book of this kind on the principles of the differential and integral calculus has been apparent for some time. It was scarcely to be expected that a mathematician of the front rank would be able to spare the necessary time from his own researches for such an undertaking, but as usual the unexpected has happened. That the Cours d'Analyse in its new form is by no means easy to read is due to the untractable nature of the material, not to any lack of skill in the author; moreover the reader will find that the difficulties that present themselves are far from being insuperable. We hope that this new edition will meet with a recognition commensurate with its deserts, and that it will do much to increase the educational value of the differential and integral calculus as instruments of logical discipline.

J. HARKNESS.

BRYN MAWR, February 10, 1894.