SOME OF THE DEVELOPMENTS IN THE THEORY OF ORDINARY DIFFERENTIAL EQUATIONS BETWEEN 1878 AND 1893.*

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SINCE the principles of the infinitesimal calculus were established, the analyst has been confronted by three problems, to wit:

The solution of algebraic equations;

The integration of algebraic differentials;

The integration of differential equations.

The history of these three problems is the same. After long and ineffectual efforts to conduct them to simpler problems, mathematicians have reconciled themselves to study these three great problems for themselves, and have been rewarded by abundant success.

For a long time the algebraist hoped and strove to solve all algebraic equations by aid of radicals. That hope has, however, been abandoned, and to-day the algebraic functions are as well known as the radicals to which it was hoped to conduct them. In the same way the integrals of algebraic differentials, which were long studied with the aim of reducing them to the elementary functions of algebra, to the logarithmic or trigonometric functions, are to-day expressed by the aid of new transcendants; and the elliptic and Abelian functions have as well defined a place in analysis as the logarithmic and trigonometric functions had less than a century ago.

Much the same thing is true of differential equations. The number of equations integrable by quadratures is extremely limited, and before the mathematician had decided to study the integrals for themselves, to study them as *functions* defined by a differential equation, all this analytical domain was but a vast *terra incognita*, which seemed to be forever interdicted to the explorer. Cauchy was the first to penetrate to the interior of this unknown region, which he did by aid of the very ingenious method which he called the calculus of limits. Many others followed him, among whom it suffices for the present to mention Fuchs, Briot and Bouquet, and Madame Kowalevski, all of whom employed his method with success.

Before taking up the class of differential equations with which these and other illustrious names have been particularly identified, it will be desirable to speak of the class of equations which, in simplicity at least, ranks first. These are the linear differential equations.

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From the time of Euler until within recent years, the only class of ordinary linear differential equations for which a general method of integration was known, was that of equations with constant coefficients. These are still the only equations whose purely external form shows that they are integrable. Among these is, of course, included Legendre's equation,

$$(ax + b)^n \frac{d^n y}{dx^n} + A(ax + b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + L(ax + b)\frac{dy}{dx} + My = 0;$$

which by the transformation $ax + b = e^t$ is at once conducted to an equation with constant coefficients.

Since the epoch-making researches of Fuchs, published in *Crelle* in volumes 66 and 68, the systematic and well-directed efforts of the most illustrious mathematicians of the age have succeeded in increasing the number of linear differential equations, for which there exist methods of integration both sure and general. The distinctive characteristics of such equations are not of a nature to be perceived at a first glance as in the cases just mentioned, but can nevertheless be recognized by aid of purely algebraic operations.

The characteristic property of a differential equation of this type is that its general integral is a *uniform* function of the independent variable.

It is by the study of the singular points of the equation that we recognize whether or not this characteristic exists. If it exists, the equation can be integrated. Its general integral is, in fact, the quotient of two synectic functions, of which one, the denominator, can be written down almost at once. This is a polynomial if the singular points are limited in number, and a holomorphic function which can be constructed by Weierstrass's method if there are an infinite number of singular points. The numerator function must be obtained by means of the differential equation. The equation is thus integrated, since its general integral is represented by one and the same function, the quotient of those just mentioned, for all values of the variable. In general this integral is a new function, but in special cases we can express it by aid of the functions already introduced into analysis. These cases are two in number. The first case, studied by Halphen, is when the general integral is uniform not only for all finite values of the variable, but for infinite values. When this is the case, the integral is a rational function. This result can only hold for equations with rational coefficients. The second case was suggested by investigations of Hermite and discovered by

Picard. Here the coefficients of the equation, in addition to being uniform functions of the variable, are doubly periodic functions having the same period, and the general integral is uniform. When this is the case the integral can be expressed in finite fo. m by means of entire polynomials, the exponential function, and Jacobi's H-function.

Starting with these three categories of linear differential equations as a basis, Halphen, in his crowned "*Mémoire sur la réduction des équations différentielles linéaires aux formes intégrables,*" proposed the following double problem:

(1) Having given a linear differential equation in the variable X and the unknown Y, to determine if there exists a substitution

$$x = \phi(X), \quad y = Y\psi(X),$$

such that, taking x for the new variable and y for the new unknown function, the transformed equation shall belong to one of the three categories:

I. Equations with constant coefficients;

II. Equations whose general integral is rational;

III. Equations whose general integral is uniform, and whose coefficients are doubly periodic functions having the same periods.

(2) Having found such a substitution, to effect the integration.

In solving a new problem the attempt is always made to simplify it by a series of suitable transformations; but there is a limit to these transformations, for in any problem there is, so to speak, something essential, which it is impossible for any transformation to alter. From this arises the importance of the general notion of invariants, which presents itself in every mathematical question. Laguerre was the first to introduce this conception of *invariant* into the theory of linear differential equations, which can thus be changed into their simplest possible forms. Brioschi has also made important contributions to the theory of the invariants of linear differential equations; but it was Halphen who, starting from his already established theory of differential invariants, has, in the memoir cited, made the most important contribution to the theory of these invariants, and employed it to solve the above problems. It is impossible to give here the results of Halphen's investigations, but two or three of his conclusions may be mentioned.

First, he says, in order that there may exist a substitution of the above form which will transform a given equation into one with constant coefficients, it is *necessary and sufficient* that its absolute invariants be constants.

Second, in order that there may exist a transformation of the above form which will transform a given equation into an equation whose general integral is rational, or into an equation with doubly periodic coefficients whose general integral is uniform, *it is necessary but not sufficient* that the invariants be connected by algebraic relations of deficiency zero or one. For the further study of this second proposition Halphen's memoir must be consulted.

Two interesting results in the case of the second problem may also be mentioned here.

(1) Having given a differential equation in which the independent variable and the coefficients are rationally expressible in terms of a parameter α:
If the ratios of the integrals regarded as functions of α have only algebraical critical points in number three at most (infinity included), and, when these points are three in number, if their orders m, n, p satisfy the condition

$$\frac{1}{m}+\frac{1}{n}+\frac{1}{p}>1;$$

we can make a change of the variable and the unknown function which will transform the equation into one whose general integral is rational.

(2) Having given a linear differential equation in which the variable and the coefficients are rationally expressible in terms of a parameter α:
If the ratios of the integrals considered as functions of α have only algebraical critical points (infinity included); and if these points are three in number and their orders satisfy the relation

$$\frac{1}{m}+\frac{1}{n}+\frac{1}{p}=1,$$

or if they are four in number and all of the second order; we can find a substitution which will change the equation into one whose coefficients are doubly periodic and whose general integral is uniform.

This last proposition is of particular interest as it gives new cases of integrability of Gauss's equation, an equation whose importance has been recognized by the ablest analysts, and upon which an enormous amount has been written. It is sufficient here to mention the names of Schwarz, Fuchs, Brioschi, and Klein. The new cases of integrability recognized by Halphen's theorem are when the equation can be integrated by aid of the elliptic functions. Halphen concludes the third chapter of his memoir, which deals with the general theory of invariants of linear equations of all orders, by indicating the method of procedure which it is desirable to follow when in the study of the critical points we substitute the consideration of the invariants for the consideration of the equation itself. In closing his introductory chapter Halphen says, "*En résumé*, I have treated in this memoir a theoretical question relative to the transformation of linear equations into other equations belonging to those types for which we know the integration to be possible. The developments indispensable to the solution of this question constitute a part, less than half, of this work. All the rest is devoted to applications which I have thought it best to multiply, even at the risk of wearying the reader. But it seems to me that in such a matter as this applications treated completely ought to dominate, and that the aim to be pursued is this: to inintegrate effectively equations which we know to be integrable."

The notion of invariants arrived at by a substitution in the case of differential equations, is only part of a much more general theory which can only be briefly alluded to here. All branches of mathematics, whether pure or applied, are intimately connected the one with the other, and notions at first restricted to a special field of research are susceptible of receiving unforeseen extensions. Such is the notion of *group*, met with to-day in every branch of mathematical research. Poincaré in his "*Notice sur Halphen*" says: "The mode of procedure of mathematical science is always the same. It studies transformations of different natures; and, to that end, it must search for that which remains constant and unaltered during these transformations. Above all, it has for aim the study of groups and for means the search for invariants. This does not appear in every case with the same distinctness, but it is always true. If we can see at first sight that projective geometry is nothing else than the theory of linear substitutions, we do not perceive so quickly that elementary geometry is conducted to the theory of orthogonal substitutions."

Since the time of Galois the theory of groups of substitutions has played a rôle of the highest importance in algebra. An analytical theory presenting close analogy to Galois's theory has been developed by Lie in his "*Theorie der Transformationsgruppen*," a most admirable account of which by Dr. Chapman has recently appeared in the BULLETIN of this Society. Lie has made the discovery of prime importance, that the search for all these groups for a given number of variables and parameters is conducted to the integration of ordinary differential equations. Lie's theory is of the highest importance in the integral calculus, the real aim of which is to integrate differential equations; it is not confined to the transformations of points, but concerns itself with the contact transformations, so important in the theory of partial differential equations. Lie has also considered the subject of continuous groups of infinite order and developed the general principles involved in the research for the invariants of differential equations.

The detailed investigations of Laguerre, Brioschi, and Halphen in the case of linear differential equations have already been mentioned. To these names must also be added that of Forsyth, who in his memoir "*Invariants, covariants, and quotient-derivatives associated with linear differential equations*" and in other papers has added important and interesting results to the general theory of the linear differential equations.

It must suffice here to merely mention a most interesting memoir by Appell on the study of the invariants and the cases of integrability:

(1) of equations of the form

$$\frac{dy}{dx} = \frac{a_0 + a_1 y + a_2 y^2 + \ldots + a_n y^n}{b_0 + b_1 y + \ldots + b_p y^p} \quad (p < n)$$

which preserve this form when we make the substitution

$$y = \eta u(x) + v(x), \quad \frac{d\xi}{dx} = \mu(x),$$

ξ being the new independent variable, and η the new unknown function;

(2) of algebraic differential equations which are homogeneous with respect to y and its derivatives and which preserve their form after the substitution

$$y = \eta u(x), \quad \frac{d\xi}{dx} = \mu(x).$$

The second part of this memoir is exceedingly interesting and suggestive, especially in its use and suggested generalization of a theorem of the author, by which, under certain conditions, the integration of an algebraic differential equation of any order and degree is conducted to the integration of a linear differential equation of an order one higher—an extension in a certain sense of the known property of Riccati's equation. The investigations of M. Roger Liouville on the invariants of non-linear differential equations of the first and second orders merit special attention, as also those of Rivereau and Painlevé, but they can only be referred to in this paper.

To speak of Poincaré's researches in differential equations is to speak of the most important discoveries in analysis of modern times. His theory of fuchsian functions is connected at once with the theory of differential equations, since it enables us to integrate linear differential equations with algebraic coefficients, and with the general theory of functions, since these transcendents present certain remarkable peculiarities which are of such a nature as to cast considerable light on the manner of being of analytical functions. These fuchsian functions are generalizations of the modular functions, studied by Hermite in his researches on the elliptic functions, which possess an infinite number of singular points distributed along a circle. It will be sufficient to mention one single property, discovered by Poincaré, of these functions, for the geometer to recognize their importance. The fuchsian functions are of two kinds: one existing in the entire plane, the other existing only in the interior of the fundamental circle. In both cases there exists an algebraic relation between two fuchsian functions which have the same group. The determination of the deficiency of this relation is of the highest importance, and has been obtained by Poincaré both by analytical processes and by aid of the geometry of position. The existence of these relations makes it possible to utilize these functions in the study of algebraic functions and algebraic curves. Thus, *we can express the co-ordinates of a point of any algebraic curve whatever as fuchsian functions, that is as uniform functions of a single parameter.*

So profound a result is sufficient in itself to show the interest and importance attaching to these new functions; but, for the present purpose, it is desirable to give at least a faint idea of their relations to the linear differential equations. To attempt to give a full idea of the importance of these functions of Poincaré's, would be, for the present writer, to attempt the impossible; indeed, a mere statement of Poincaré's results produces an impression of exaggeration.

Since the time of Cauchy, mathematicians, recognizing the enormous difficulties and complications of the problem, have not attempted to study the nature of the integrals of differential equations, ordinary or partial, for all values of the variable,—that is, throughout the plane,—but have confined themselves to the investigation of the properties of these integrals in the neighborhood of certain given points. They thus perceived that these properties are very different according as we are concerned with an ordinary point or a singular point. The researches of Briot and Bouquet are too well known to need more than mention, and since their appearance most important additions to the theory of non-linear equations have been made, notably by Fuchs, Poincaré, Picard, and more recently by Painlevé. But the study of the integrals of differential equations in the neighborhood of a given point, whatever may be its utility from the point of view of numerical calculation, can only be regarded as a first step in this study. These developments, which hold only in a very limited region, do not teach us concerning the integrals of differential equations that which the θ-functions teach us concerning the integrals of algebraic differentials : they cannot be considered as true integrations.

They must then be taken as a point of departure for a more profound study of the integrals of differential equations, in which it is proposed to free ourselves from the restrictions of these "limited regions," where, to use a phrase of Poincaré's, "*on s'était systématiquement cantonné*," and to follow the integral as the variable moves throughout the entire plane.

This generalized study can be entered upon from two points of view.

(1) We may propose to express the integrals by developments which *always* hold, and are no longer limited to a particular domain. This leads to the introduction of new transcendants into the theory ; but this introduction is necessary in any case, for the functions which have previously been introduced into analysis will only permit us to integrate a very small number of differential equations.

(2) This method of integration, though affording us a knowledge of the properties of the differential equations from the point of view of the theory of functions, is not in itself sufficient if, for example, we wish to apply the equations to questions of mechanics or physics. These developments would not readily teach us, for example, if the function was one which continually increased, if it oscillated between certain limits, or if it increased beyond all limit. In other words, if the function is considered as defining a plane curve we would not know the general form of the curve. In certain applications all these questions are of as much importance as the numerical calculation. We have here then a new problem for solution. These are the two problems that Poincaré proposed to himself and undertook and succeeded in the task of solving,—truly a task for a giant of intellect.

"Desiring," to quote Poincaré, "to express the integrals of differential equations by the aid of series *always* convergent, I was naturally led first to attack the linear differential equations. These equations, in fact, which have been during recent years the object of the investigations of Fuchs, Thomé, Frobenius, Schwarz, Klein, and Halphen, were the best known of all. We had for a long time possessed the developments of their integrals in the neighborhood of a given point, and in a quite large number of cases had succeeded in completely integrating them by aid of functions already known.

It was then in commencing my studies at this point that I had the most chances of arriving at a result.

"But it was further necessary to make some hypothesis concerning the coefficients of the equations which I wished to study. If, in fact, I had taken for coefficients *any functions whatever,* I would equally have obtained for integrals *any functions whatever,* and consequently would not have been able to say anything precise on the subject of the nature of these integrals, which was my aim. I was thus led to examine the linear equations with rational and with algebraic coefficients."

The following is Poincaré's classification of these linear equations, the one which, from the point of view of the proposed problem, is the most natural.

Let y be an integral of a linear equation of order n with rational coefficients. Write

$$z = e^{\int\lambda dx}\left(F_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + F_{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \ldots + F_1\frac{dy}{dx} + F_0 y\right),$$

where λ and the F's are rational functions of x. It is clear that z, like y, will satisfy a linear equation of order n with rational coefficients. Poincaré says that these equations belong to the same *family*. In fact, it is easily seen that the knowledge of the properties of the function y involves that of the properties of the function z. In each family there are an infinity of different equations, but certain functions of the coefficients have the same value for equations of the same family. In other words, there are *invariants* which remain unaltered by the above substitution. These invariants are not the same as those of Halphen for linear equations. These latter arise from a transformation which consists in replacing x by *any function whatever* of x' and multiplying y by *any other function whatever* of x'. On the contrary, the functions entering into Poincaré's substitution are not any whatever, but are rational. This shows well the difference between Halphen's investigations and those of Poincaré. Halphen sought above all to find the relations between different integrals, and could thus with impunity introduce any functions whatever into his calculations. Poincaré, on the other hand, sought to study the nature of the integral itself; this nature would manifestly be altered if the integral were multiplied by an arbitrary function, as is the case in Halphen's work.

This intimate study of the nature of the integrals can only be made by the introduction of new transcendents. These new transcendents have a close analogy to the elliptic functions, which is not surprising, as Poincaré was attempting to do for differential equations with algebraic coefficients what had already been done by aid of the elliptic and Abelian theta-functions for the integration of algebraic differentials.

This analogy with the elliptic functions served Poincaré as a guide in all his researches. The elliptic functions are uniform functions which are unaltered when the variable is increased by certain periods. This notion is so useful in mathematical analysis that mathematicians had long seen the desirability of generalizing it by seeking uniform functions of a single variable x, which would remain unaltered when this variable was subjected to certain transformations. These transformations cannot be chosen arbitrarily, but must evidently form a group: and further, this group must contain no infinitesimal transformation; that is, x must not vary by an infinitesimal amount. For, if this were so, on repeating this transformation indefinitely, x would vary continuously; and our uniform function, remaining unaltered when the variable changed continuously, would necessarily reduce to a constant. In other words, the group of transformations must be discontinuous.

In elliptic functions the transformations of the group consist in adding constants. The functions are studied by dividing the plane into an infinite number of parallelograms of periods. All the parallelograms are obtained by transforming one of them by the different substitutions of the group; so that a knowledge of the function in one of these parallelograms involves a knowledge of it throughout the entire plane.

So if we consider a more complicated discontinuous group generated by a transcendent of higher order, we can divide the plane (or the part of the plane in which the function exists) into an infinity of regions, or curvilinear polygons, in such a way that we can obtain all these regions in applying to one of them the different transformations of the group. The knowledge of the function in the interior of one of these curvilinear polygons will involve the knowledge of the function for all possible values of the variable.

In the elliptic functions, considering the integrals of "the first kind," we regard the variable x by the process of inversion as a function of the integral. The function thus defined is uniform and doubly periodic. So, considering a linear equation of the second order, and by a species of inversion, Poincaré regards the variable x not as a function of an integral but as a function of the ratio of the two integrals of the equation. In certain cases the function thus defined is uniform, and then remains unchanged by an infinity of linear substitutions changing z into

$$\frac{\alpha z + \beta}{\gamma z + \delta}.$$

When this is the case, the group formed by these substitutions must be discontinuous, and the curvilinear polygons referred to must be limited by arcs of circles. Poincaré first supposes the coefficients of the substitutions

$$\left(z, \quad \frac{\alpha z + \beta}{\gamma z + \delta}\right)$$

to be real, or, what comes to the same thing, supposes that these substitutions do not alter a certain circle which he calls the "*fundamental circle*." In this case the arcs of circles which serve as sides of the polygons cut the fundamental circle orthogonally.

What is then the condition that a group generated by a given curvilinear polygon shall be discontinuous? Poincaré finds this condition, and then constructs the discontinuous groups formed by substitutions which leave the fundamental circle unaltered, and which he calls fuchsian groups. Here an important problem now presents itself: having given a fuchsian group, do there exist functions which are unaltered by the substitutions of this group? In answering this question Poincaré appeals again to the analogy with the elliptic functions. These functions can be regarded as the quotients of two θ-series. These auxiliary transcendents are not only uniform, but are also entire functions; they are not doubly periodic, but reproduce themselves multiplied by an exponential when the variable is increased by a period. So in the new case the fuchsian functions are expressed as the quotient of two "*theta-fuchsian*" series, which are finite and uniform, perfectly analogous to the θ-functions, and reproduce themselves multiplied by a simple factor when we apply to the variable one of the substitutions of the group.

In order to complete the analogy with the elliptic functions, it was necessary that the other properties of these functions, such as addition, multiplication, and transformation, should be extended to the new transcendents. The theory of transformation is immediately generalized—always with this difference, that the group of fuchsian functions being very much more complicated than that of the elliptic functions, the cases to be considered are much more numerous and varied. A point of great interest in the extension of this theory of transformation is the new light thrown upon the reduction of the Abelian integrals. The theory of addition cannot be extended to all the fuchsian functions; it is only possible in one particular case and for one special class of these transcendents. The question need not, however, be entered into here.

In extending the linear substitutions to the case where the coefficients are no longer restricted to be real, but are arbitrary, Poincaré arrives at the discontinuous groups which he calls kleinean, and so to a new class of functions, the kleinean functions, which are perfectly analogous to the fuchsian functions. The only difference which it is necessary to mention is that which arises from the form of the region inside of which these new functions exist. This region instead of being a circle is limited by a non-analytical curve which has no determinate radius of curvature; in other cases the region is limited by an infinity of circumferences. Another class of functions can only be mentioned. These are the zeta-fuchsian functions, which play the same part in the integration of the fuchsian functions that the zeta-functions play in the integration of the elliptic functions.

In concluding this very imperfect sketch of what Poincaré has done for the integration of linear differential equations, we can say that he has shown how to express the integrals of linear equations with algebraic coefficients by aid of these new transcendents in the same way as the integrals of algebraic differentials were previously expressed by aid of the Abelian functions. Further, these latter integrals are themselves susceptible of being obtained by aid of the fuchsian functions; and we thus have for them expressions entirely different from those given by the θ-functions of several variables.

Poincaré's further investigations on the integration of linear differential equations by aid of algebraic and Abelian functions, his work on non-linear equations (with one exception to be referred to later), on curves defined by differential equations, on irregular integrals, on partial differential equations, and on the differential equations of celestial mechanics, must be passed over. Enough has been said, however, to show the great value of his work. In leaving the subject of linear differential equations a mere mention will have to suffice of two interesting papers which have recently appeared: one by Appell, on linear differential equations transformable into themselves by change of the function and the variable; and one by Helge von Koch, on infinite determinants and linear differential equations.

The linear differential equations possess one remarkable property,—the singular points are the same for all the integrals In the case of equations whose coefficients are entire polynomials in x, these singular points are the values of x which annul the first coefficient. It is upon this circumstance that the method of integration of these equations by the zeta-fuchsian functions is founded. Non-linear equations do not in general possess this property. This has led to important investigations by Fuchs and Poincaré with the end of ascertaining whether or not there existed other classes of differential equations for which all the particular integrals had the same singular points. The researches of these mathematicians were confined to equations of the first order.

Fuchs found the necessary and sufficient conditions that an integral of the equation

$$f(x, y, y') = 0$$

where y, y' enter algebraically, shall have only fixed critical points x_i. These conditions once satisfied, if we wish the general integral to be uniform, it remains to express that these points x_i, which are known, are not critical points of the integrals. Poincaré resuming the question, arrived at conclusions as interesting as unexpected. He found that when the critical points of the equation

$$f(x, y, y') = 0$$

are fixed this equation can either be integrated algebraically, or by a quadrature, or can be conducted to a Riccati's equation.

Picard has undertaken the generalization of the work of Fuchs and Poincaré to the case of differential equations of the second order. He says: "It would seem at a first glance that the extension of the reasonings employed in the case of equations of the first order would be easy, but this is not the case. We may indeed commence by following the methods of reasoning employed by Poincaré; the end of his reasoning is unfortunately not applicable. We find ourselves always in presence of the same fact: A certain bi-uniform transformation is not necessarily bi-rational, and it is this fact which changes throughout the entire character of this theory." Picard confines himself to the case of bi-rational transformations. The difficulties in the way of a study of non-linear differential equations of the second order have caused this subject to remain as yet almost untouched; indeed, though in recent years much has been done, the non-linear equations of the first order still afford a vast field for research.

A problem of the highest importance connected with the differential equations of the first order and degree is to determine in any case whether or not the equation is integrable algebraically. To solve this problem it is manifestly sufficient to find a superior limit for the degree of the integral; after that the only operations remaining to be performed would be purely algebraical. This is a problem on the solution of which, it would seem, mathematicians might have been tempted to bestow much labor; nevertheless, they have concerned themselves very little with it. Indeed, from the time of Darboux's masterly memoir in 1878, the problem was entirely neglected, until the Académie des Sciences, recognizing the importance of a more profound study of differential equations of the first order and degree, proposed as a subject of competition for the *Grand Prix des Sciences Mathématiques* for 1890 the following:

"To perfect in some important point the theory of the differential equations of the first order and first degree."

The outcome of the competition has been two important memoirs: one by M. Painlevé, to which was awarded the Grand Prize; and the other by M. Autonne, who received honorable mention.

M. Autonne takes for point of departure a geometrical interpretation of which every differential equation of the first order is susceptible. He shows that such an equation can be considered as giving curves situated on a certain algebraic surface, and of which the tangents belong to a certain linear complex. The surface is unicursal if the equation of the first order is at the same time of the first degree. Taking then a unicursal surface, the author forms the equation which corresponds to it, and which he calls "*réglementaire.*" His work then consists of a study of the equations of the first order and degree considered as "*réglementaires.*" Autonne makes a classification of the critical points of the equation. It will be sufficient to mention, in addition to the ordinary critical points forming the general case, those which he calls "*dicritical.*" These are points through which pass an infinite number of simple branches of integrals having an arbitrary tangent.

Painlevé's researches constitute one of the most important contributions ever made to the theory of the non-linear differential equations of the first order and degree. He considers any differential equation whatever of the first order where the function and its derivative figure algebraically, and makes at first an important distinction between the fixed and the movable critical points of the integrals,—by movable critical points meaning those which are susceptible of changes of position as the constant of integration changes. Such points cannot be points of indetermination for the integral. The utility of making this remark explicitly is seen if it is noticed that equations of an order higher than the first can have movable singular points. As an illustration, take the equation

$$(yy'' - y_2'^2)^2 + 4yy'^3 = 0,$$

whose general integral is

$$y = C_1 e^{\frac{1}{x - C_2}};$$

the essential singularities of the integrals depend evidently on the constant C_2.

Painlevé draws a system of cuts in the plane of the independent variable, which prevent this variable from turning round the fixed critical points, and then studies the equations for which the integrals take only a limited number of values around the movable critical points. Under this hypothesis we can conceive the general integral put into a form which will bring into evidence a class c of algebraic curves associated with the given equation.

Every curve of this class is a rational transform of the curve represented by the differential equation when for any fixed value whatever of the variable we regard the function and its derivative as the co-ordinates. If the deficiency of the curves c is greater than unity, we can by algebraic operations determine the class, and the integral itself can be obtained algebraically. If the curves c are of deficiency one, it may be necessary, in order to find the integral, to obtain a solution of a linear equation, and to ascertain if a certain Abelian integral has only two periods. Only the case of deficiency zero escapes this method; this circumstance necessarily presents itself when the equation is of the first degree with respect to the derivative.

Painlevé's methods can be applied in the attempt to ascertain whether or not the integrals of a given differential equation are algebraic, or can take only a limited number of values in the plane. The distinction between fixed and movable critical points permits the separation of this question into two parts, and in some cases the solution of this problem can be arrived at,—a solution which in all its generality will doubtless not soon be found.

The equalities and inequalities added by Autonne and Painlevé to those of Darboux constitute a very important progress in the solution of the problem of finding out whether or not a differential equation of the first order and degree is algebraically integrable; but, as Poincaré has pointed out in a recent paper in the *Rendiconti del Circolo Matematico di Palermo,* much remains to be done. Suppose, in fact, that the general integral is written

$$F = \text{const.},$$

where F is a rational function. Another form of the general integral will be given by equating to a constant any entire polynomial whatever in F. It follows as a consequence of this that the superior limit of the degree of the general algebraic integral cannot be found, at least unless some means can be found of expressing in the inequalities that this integral is irreducible.

Painlevé himself clearly perceived this difficulty, but was unable to overcome it. He was unable to solve the problem in all its generality, but only to show that in a certain number of cases the integral could not be algebraic. As already stated, Painlevé's problem was to determine whether a given differential equation admitted an algebraic integral of a given deficiency. In the paper referred to, Poincaré gives a formula which contains the complete solution of Painlevé's problem whenever the dimension of the equation exceeds 4. He has also demonstrated some properties of equations which are integrable algebraically. Such results, he says, may not for the moment possess any great value, but they may acquire one the moment it is known whether these properties can be extended to non-integrable equations, or if they are not always true for these equations. In the first case we would have a general theorem applicable to all differential equations, and in the second case we would have a criterion which would enable us to demonstrate that the equations of certain categories were non-integrable.

In the subject of partial differential equations most important researches have been made during the past ten years, notably by Goursat, Poincaré, Picard, and Darboux. Indeed, the latter's treatise on the theory of surfaces might, from a certain point of view, be regarded as a treatise on these equations, particularly the equations of the second order. Picard has incorporated some of his more important results in the first fascicule of the second volume of his *Traité d'Analyse.* There also, among other methods for the proof of the existence of an integral of a differential equation, he gives his own elegant method of proof by successive approximations, an English translation of which by Dr. Fiske has been published in the BULLETIN of this Society. Picard has made some interesting extensions of this method, particularly to partial differential equations. The whole subject, however, of the recent developments of the theory of these equations and their applications to geometry must be reserved for another paper.

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