outside of the lecture room ? What engineer would use foot poundal for example ? The nomenclature is also redundant. A single instance will suffice. Shall we say vis-viva, living force, or kinetic energy ? All three are used to denote the same thing to the mystification of the beginner. All three can be found in text books of recent date. To my mind there is no doubt but that kinetic energy is the proper term.

Now, the confusion, deficiency, and redundancy being granted, what can be done ? No one writer can do much to effect a change. But an association such as the *New York Mathematical Society* can do much. Expressions of opinion through the pages of this journal would probably lead to some more definite understanding than now exists. At least some of the more glaring absurdities and contradictions of our present system might be abated. Besides, it might tend to curb the ambition of writers to introduce ill-considered terms such as "heaviness" or "centre of weight" for centre of gravity and the like.

UNION COLLEGE, 1891, *October* 10.

A TREATISE ON LINEAR DIFFERENTIAL EQUATIONS. Vol. I. Equations with uniform coefficients. By THOMAS CRAIG, Ph.D. New York; John Wiley & Sons, 1889. 8vo, pp. ix. + 516.

THE appearance of Fuchs's two memoirs in 1866 and 1868 respectively, gave an impetus to research on linear differential equations which has resulted in the development of an enormous literature on the subject, consisting of articles and memoirs scattered through mathematical journals and the proceedings of learned societies. The systematization and presentation in a body of the principal methods and results developed in these isolated papers, is the work which has been undertaken by Professor Craig, and which has successfully issued in the first volume of the most advanced treatise on pure mathematics ever published by an American author. Whilst the presentation of the subject as a whole must prove of advantage to those few mathematicians who have access to the memoirs whence it draws, upon the many to whom the original sources are not open it confers an inestimable boon. To the English-reading student further it manifests in his own language the substance of what is for the most part in the original in French or German. Praise is due the author for the scrupulous care with which he credits every writer quoted, and for the fulness of his references, which give an added value to the volume. A glance at these references cannot fail to impress upon the reader a sense of the overwhelming influence which the continental element has had in shaping the development of modern differential equations. In fact, an analysis shows that of the sixty-odd names quoted in the volume more than three-fourths divide themselves about equally between the French and Germans, and of the remainder some eight may be claimed by the English speaking peoples: so that if this showing in relation to the populations of the countries concerned could be fairly considered as furnishing a criterion relative to the generality of interest manifested among the several peoples in the development of the subject, such interest in America and England as compared with that in France and Germany might be averaged as 1 to 7. The dropping of the average in the comparison, it may be frankly owned, would not advantage the showing of America.

The reader in his progress through this treatise will constantly have to do with the modern theory of functions, and will meet with some simple applications of the theory of substitutions. Both of these departments, with their numerous applications and possibilities of further development, offer a field whose successful cultivation on the continent shows a productive power giving as yet no sign of exhaustion. Professor Craig's book will have accomplished a useful mission if it helps to awaken American students to a sense of the work that is being done in Europe, and, as a consequence, rouses them to a realization of what is being left undone in America. There seem, however, at present to be definite tendencies making for the elevation of mathematics in America, and it may not perhaps be idle to indulge a hope that America will yet contribute in a fitting proportion to the development of the science. The preliminary knowledge of the theory of functions necessary to the reading of Professor Craig's book may be obtained from Hermite's *Cours*. To the student who desires an acquaintance with the theory of substitutions one can recommend Netto's *Substitutionentheorie* and Serret's *Cours d'Algebre Superieure*, though so far as is necessary for understanding the applications of the latter theory in the volume under consideration, a very partial reading of its treatment in either of the works mentioned will prove sufficient, and, in fact, a few words of explanation from one familiar with the substitution notation would probably suffice. The American student of mathematics who acquires a knowledge of these branches will in general do so by his own unaided efforts, for courses in them are offered by but a small number of our universities, and further, as regards unassisted study, it may unfortunately be said that few of our colleges and universities give a course in mathematics whose discipline prepares a man for such study. The fault lies perhaps not so much with the higher institutions of learning as with the preparatory and high schools, into whose hands our potential young mathematicians first fall, and which as a general rule allot to the study of algebra and geometry a time utterly inadequate to the laying of a basis on which the college can satisfactorily build. On the other hand, almost all our college professors, among whom we find, of course, the great majority of our mathematicians, are overworked. Teaching absorbs the energy and spontaneity which should be spent upon private study and research. For the latter scanty allowance is made, except in a few of our larger universities, conspicuous among which is that university in which the author of our treatise is a teacher. The lack of stimulus and encouragement due to the isolation in which the American mathematical professor has been wont to live, may (it is not an unreasonable anticipation) be remedied in some degree by the founding of a mathematical society of national scope with the publication of a bulletin. Thus may be fostered among American mathematicians a fellow interest in their science, to illustrate the advantages of which we might cite the subject of the work before us, which has been developed since the publication of Fuchs's memoirs, only by the cross-working of scores of European mathematicians.

Before the appearance of these two memoirs the only general class of linear differential equations for which a solution had been found was that in which all the coefficients are constant, but with the application of the modern theory of functions a new field opened up. In this theory the critical points of a function play an all-important role, and, as can be readily shown in the case of the equation which constitutes the theme of the volume under review, the critical points of the integrals of the equation are included among those of its coefficients. This property evidently gives us some hold upon the integrals and is, when combined with the fact that the general integral is a linear function of the particular integrals, more fruitful of results than would readily be anticipated, results of which but a few can here be hinted at.

The work opens with a recapitulation of the general properties of linear differential equations, followed by an extended modern treatment of the equation with constant coefficients. It then takes up the theory of the differential equation

$$(1)\ P(y) = \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1}y}{dx^{n-1}} + p_2 \frac{d^{n-2}y}{dx^{n-2}} \cdots\cdots + p_n y = 0.$$

where the coefficients $p_1, p_2, \ldots, p_n$ are uniform functions of x, having only poles as critical points. Let $y_1, y_2, \ldots y_n$ denote a system of fundamental integrals of (1). If now the imaginary variable x make the circuit of a critical point in the plane, returning by any path to its point of departure, the coefficients, since they are uniform, will return into themselves, and the equation will be unaltered. Any integral of the original equation, then, necessarily remains such, and can at most have transformed into a linear function of the n fundamental integrals. It is now shown that among such transformed integrals, there will be at least one which will transform into itself multiplied by some constant s which is determined as the root of an equation of the nth degree in s called in reference to the critical point in question, the characteristic equation for the system of fundamental integrals $y_1, y_2, \ldots, y_n$.

There will be as many such integrals as there are solutions to the characteristic equation; and, in fact, corresponding to a λ-multiple root s_1 of this equation there will be a group of λ integrals $u_1, u_2, \ldots, u_\lambda$ which, when the variable x completes the closed circuit, may respectively be shown to transform into

$$\begin{aligned}
&s_1 u_1, \\
&s_{21} u_1 + s_1 u_2, \\
&s_{31} u_1 + s_{32} u_2 + s_1 u_3, \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&s_{\lambda 1} u_1 + s_{\lambda 2} u_2 + \ldots + s_{\lambda, \lambda - 1} u_{\lambda - 1} + s_1 u_\lambda.
\end{aligned}$$

where the coefficients s are all constants, and the aggregate of such groups corresponding to the different roots of the characteristic equation will constitute a system of fundamental integrals of the differential equation. The theory is given for the point $x = 0$ considered as the typical critical point, the reasoning for any other critical point a being obtained by substituting $(x-a)$ for x wherever it may appear in our formulæ.

The group of integrals given above are now shown to be of the following forms:

$$(2) \quad \begin{cases}
u_1 = x^{r_1} \varphi_{11} \\
u_2 = x^{r_1} \{\varphi_{21} + \varphi_{22} \log x\} \\
u_3 = x^{r_1} \{\varphi_{31} + \varphi_{32} \log x + \varphi_{33} \log^2 x\} \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
u_\lambda = x^{r_1} \{\varphi_{\lambda_1} + \varphi_{\lambda_2} \log x + \ldots\ldots + \varphi_{\lambda\lambda} \log^{\lambda-1} x\}
\end{cases}$$

where the φ's are uniform in the region of our critical point

$x = 0$, and such that anyone of them can be expressed in terms of those whose second subscript is 1; $\varphi_{11}, \varphi_{22} \ldots\ldots, \varphi_{\lambda\lambda}$ differing from one another only by constant factors and $2\pi i r_1$ being equal to $\log s$; we find that this group (2) may be replaced by a number of sub-groups possessing precisely the properties just enumerated, and can further show that the transformation effected by a circuit of the critical point may be represented thus:

$$S \equiv \left| \begin{array}{l} y_1, \ y_2, \ \ldots; \ s_1 y_1, \ s_1(y_2 + y_1), \ \ldots, \ s_1(y_\alpha + y_{\alpha-1}) \\ y'_1, \ y'_2, \ \ldots; \ s_1 y'_1, \ s_1(y'_2 + y'_1), \ \ldots, \ s_1(y_{\alpha'} + y_{\alpha'-1}) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ z_1, \ z_2, \ \ldots; \ s_2 z_1, \ s_2(z_2 + z_1), \ \ldots, \ s_2(z_\beta + z_{\beta-1}) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array} \right|$$

the interpretation of this notation being that a certain system of n independent integrals of our equation represented by the symbols $y_1, y_2 \ldots. y'_1 \ldots. z_1 \ldots.$ on the left, are by circuit of the critical point transformed into the expressions corresponding in position on the right, where $s_1, s_2, \ldots.$ are solutions of the *characteristic* equation. The aggregate of transformations thus effected is indicated by the letter S and is called the substitution for the point in question. As presented here it is said to be in its canonical form. There will be such a substitution corresponding to a single circuit of any critical point, to a multiple circuit of the same, or to a circuit including any combination of critical points, all substitutions, by the way, being reducible to successive applications of the substitutions for different individual points.

The aggregate of all possible substitutions is defined as the *group* of the equation. In a later chapter the author by the aid of the canonical form just given goes into the investigation of what are called function-groups, these being groups of functions which under all possible applications of a substitution-group transform into one another. The integrals of our equation (1) evidently constitute such a group and include, it may be, smaller function-groups formed by the linear functions of linearly independent integrals less than n in number. With the consideration of these the chapter just referred to concerns itself.

Reverting now to formulæ (2), if all the φ's entering into any one of the integrals u contain only finite negative powers of x, the integral is called *regular* in the region of the point $x = 0$, and with proper choice of r can be written

$$(3) \qquad F \equiv x^r \left\{ \varphi_0 + \varphi_1 \log x + \ldots. + \varphi_k \log^k x \right\},$$

where the φ's are uniform, and $x^{-r}F$ becomes infinite for $x = 0$ in the same manner as $\alpha + \beta \log x + \ldots\ldots\ldots + \lambda \log^k x$, α, β, ... being constants. In order that equation (1) should have a system of linearly independent regular integrals in the region of the point $x = 0$, it is shown to be necessary and sufficient that every coefficient p_j shall have $x = 0$ as an ordinary point or a pole of multiplicity not greater than j. Denoting by w_i the degree of x in the denominator of p_i, the value of i for which $w_i + n - i \equiv g$ is a maximum is called the *characteristic index* of equation (1), and by substitution in the differential quantic $P(y)$ of x^ρ for y, we will find that $x^{-\rho}P(x^\rho)$ developed in ascending powers of x has as its first term $G(\rho)\, x^{-g}$ where $G(\rho)$ is an integral function of ρ of degree $n - i \equiv \gamma$. $G(\rho) = 0$ is called the *indicial equation*, and it may be shown that the number of linearly independent regular integrals of (1) is not greater than the degree of this equation. The conditions that it should be equal to this degree are also determined, and in particular its degree is observed to be equal to n when all the integrals are regular. The exponent r in (3), where F is supposed to be a regular integral, is given by the indicial equation; and the coefficients of the φ's developed in positive powers of x are determined by substitution of F in the differential equation.

An extended application of the general theory is made to differential equations of the second order, particularly to the equation which has all its integrals regular and possesses but three critical points. This equation is shown to be transformable to one in which the critical points are 0, 1, ∞, an equation of which the hypergeometric series $F(\alpha, \beta, \gamma, x)$ is an integral. A complete translation of Goursat's memoir on this equation is embodied in the work, filling some 150 pages. An exhaustive discussion is given of its twenty-four integrals, which divided into six groups of four each, are connected by some twenty linear relations between integrals selected from the six groups taken three at a time. The portions of the plane in which the several integrals have a meaning are also indicated. An investigation is made of the transformations admitted by the series when all three quantities α, β, γ are not arbitrary; and an extended list of such transformations, with formulæ derived therefrom, is given. The theory of irreducible equations is briefly touched upon; as is also, at greater length, the theory of the decomposition of a linear differential equation into prime factors, with its application in the case of equations possessing regular integrals.

In equation (1) we can by a simple transformation readily get rid of its second term; and, as is shown in one of the later chapters of the book, by a transformation $z = \varphi(x)$, $y = z'^{-\frac{1}{2}(n-1)}u$, where the form of z is dependent on a differential equation of second order, we may still further rid ourselves of its third term, the equation so reduced being said to be in its *canonical form.* There are also certain *associate equations* $(n - 2)$ in number, the solutions of each of which consist in a set of variables dependent upon the integrals of equation (1) and possessing relative to the transformation mentioned, the invariantive property of returning into themselves multiplied by a power of z', among these equations being found the well-known equation of the n^{th} order on which depends the determination of an integrating factor for (1).

The volume concludes with a short chapter on equations with uniform doubly-periodic coefficients, a subject which the author expresses his intention of resuming in his second volume. Supposing w and w' to be the periods of our coefficients, by the substitution of $x + w$ or $x + w'$ for x, they will remain unaltered and the integrals will transform into linear functions of one another. By analogy the general theory already given suggests that the characteristic equations corresponding to these substitutions may give us constants s and s', by which the respective transformations multiply some integral u. When the general integral happens to be uniform such proves to be the case, there being at least one integral u which by the substitutions $x + w$ and $x + w'$ for x respectively transforms into $s\,u$ and $s'u$, and for the determination of such integrals, as also of the other integrals of the equation, methods are given. J. C. FIELDS.

NOTES.

AT the meeting of the NEW YORK MATHEMATICAL SOCIETY held Saturday afternoon, October 3d, at half-past three o'clock, the Council announced that Professor Henry B. Fine had been appointed to fill the vacancy in their body. The following persons having been duly nominated, and being recommended by the Council, were elected to membership: Professor Thomas Craig, Johns Hopkins University; Dr. A. V. Lane, Dallas, Texas; Professor L. A. Wait, Cornell University; Professor George Egbert Fisher, University of Pennsylvania; Mr. William H. Metzler, Clark University; Professor Ellen Hayes, Wellesley College; Professor George A. Miller, Eureka College; Mr. Charles Nelson Jones, Milwaukee, Wisconsin; Dr. J. Woodbridge Davis, New York; Mr. Charles H. Rockwell, Tarrytown, N. Y.; Professor J. Burkitt Webb, Stevens Institute of Technology.