

Physical aspects of quantum sheaf cohomology for deformations of tangent bundles of toric varieties

Ron Donagi¹, Josh Guffin¹, Sheldon Katz² and Eric Sharpe³

¹Department of Mathematics, University of Pennsylvania, David
Rittenhouse Laboratory, 209 South 33rd Street, Philadelphia, PA
19104-6395, USA
donagi@math.upenn.edu
guffin@math.upenn.edu

²Department of Mathematics, University of Illinois, 1409 W. Green Street,
Urbana, IL 61801, USA
katz@math.uiuc.edu

³Physics Department, Robeson Hall (0435), Virginia Tech, Blacksburg, VA
24061, USA
ersharpe@vt.edu

Abstract

In this paper, we will outline computations of quantum sheaf cohomology for deformations of tangent bundles of toric varieties, for those deformations describable as deformations of toric Euler sequences. Quantum sheaf cohomology is a heterotic analogue of quantum cohomology, a

quantum deformation of the classical product on sheaf cohomology groups, that computes nonperturbative corrections to analogues of $\overline{27}^3$ couplings in heterotic string compactifications. Previous computations have relied on either physics-based gauged linear sigma model (GLSM) techniques or computation-intensive brute-force Čech cohomology techniques. This paper describes methods for greatly simplifying mathematical computations, and derives more general results than previously obtainable with GLSM techniques. We will outline recent results (rigorous proofs will appear elsewhere).

CONTENTS

1	Introduction	1257
2	General procedure and definitions	1258
3	Example: projective space	1265
4	Example: product of projective spaces	1269
	4.1 Classical cup products on $\mathbf{P}^1 \times \mathbf{P}^1$	1270
	4.2 Classical cup products on $\mathbf{P}^n \times \mathbf{P}^m$	1272
	4.3 Quantum sheaf cohomology ring on $\mathbf{P}^1 \times \mathbf{P}^1$	1275
5	Example: Hirzebruch surface	1278
6	General result	1286
	6.1 Result	1286
	6.2 Comparison to McOrist–Melnikov’s results	1290
7	Conclusions	1295
8	Acknowledgments	1295
	Appendix A. GLSM derivation of four-Fermi terms	1296
	References	1299

1 Introduction

This paper is concerned with computing quantum sheaf cohomology rings, an analogue of quantum cohomology rings for heterotic strings.

Quantum cohomology describes the operator product rings in A model topological field theories. Those operator product rings are deformations of the classical cohomology rings, and so are called “quantum cohomology” rings. The deformations encode information about minimal-area surfaces, and so quantum cohomology played an important role in the enumerative geometry revolution that swept through algebraic geometry starting in the early 1990s, and continues in various forms to this day.

Quantum sheaf cohomology computes analogous invariants of pairs consisting of spaces X together with vector bundles $\mathcal{E} \rightarrow X$ satisfying the conditions

$$\Lambda^{\text{top}} \mathcal{E}^* \cong K_X, \quad \text{ch}_2(\mathcal{E}) = \text{ch}_2(TX).$$

Such pairs define the “A/2 model,” a heterotic generalization of the A model. An analogue of quantum cohomology for the A/2 model was originally defined in [1] (motivated by physics considerations in [2]), and describes a deformation of the product structure on sheaf cohomology, for which reason this deformation has been named “quantum sheaf cohomology.” Much as in ordinary quantum cohomology, the deformation in question revolves around enumerative properties of X — specifically, one computes sheaf cohomology of induced sheaves over a moduli space of curves in X , corresponding physically to nonperturbative corrections to correlation functions of charged fields.

Quantum sheaf cohomology and related notions have been further developed in a variety of recent papers including e.g., [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

In this paper we shall outline general results for quantum sheaf cohomology for X a compact toric variety and \mathcal{E} a deformation of the tangent bundle of X , described as a deformation of the toric Euler sequence. In particular, in the past such computations have been done with either physics-based gauged linear sigma model (GLSM) techniques (which so far have not been amenable to studying nonlinear deformations), or math-based computation-intensive brute-force Cech cohomology computations. One of the innovations of this paper and [20] are a set of new ideas to radically simplify mathematics computations, which we use to obtain results of greater generality than previously obtainable with GLSM techniques. Utilizing those methods, we

find, for example, that quantum sheaf cohomology rings, at least in these cases, are independent of nonlinear deformations, a result previously conjectured in [11, 13]. Detailed proofs are left to [20].

We begin in Section 2 by describing the A/2 model (a holomorphic field theory), and outline the correlation function computations in that theory, first at a formal level, then describing generalities of linear sigma model (LSM) compactifications and induced sheaves over moduli spaces of curves. In Section 3 we begin by computing the quantum sheaf cohomology of a projective space. Since the tangent bundle of a projective space is rigid, the result will automatically match the ordinary quantum cohomology ring, but this is a useful warm-up exercise and demonstration of some of the technology we are introducing that simplify general quantum sheaf cohomology computations. In Section 4, we apply these ideas to compute quantum sheaf cohomology on a product of projective spaces. Briefly, quantum sheaf cohomology reduces to a classical sheaf cohomology computation over the LSM moduli spaces, and for a product of projective spaces, the LSM moduli spaces are again a product of projective spaces, so we work through classical sheaf cohomology for products of general projective spaces, then apply those results to quickly compute quantum sheaf cohomology for a deformation of the tangent bundle on $\mathbf{P}^1 \times \mathbf{P}^1$. Projective spaces are a bit simple, so in Section 5 we compute quantum sheaf cohomology for a deformation of the tangent bundle on a Hirzebruch surface, which allows us to tackle issues such as nonlinear deformations and four-Fermi interaction terms. In Section 6 we describe general results (derived in detail in [20]). In Appendix A we derive an ansatz for four-Fermi terms from GLSMs, that is used both in this paper and in [20].

2 General procedure and definitions

First, let us briefly review the A/2 model. Recall that on the (2, 2) locus, the A model topological field theory is a twist of the (2, 2) supersymmetric NLSM (nonlinear sigma model)

$$\frac{1}{\alpha'} \int_{\Sigma} d^2z \left((g_{\mu\nu} + iB_{\mu\nu}) \partial\phi^{\mu} \bar{\partial}\phi^{\nu} + \frac{i}{2} g_{\mu\nu} \psi_{+}^{\mu} D_{\bar{z}} \psi_{+}^{\nu} + \frac{i}{2} g_{\mu\nu} \psi_{-}^{\mu} D_z \psi_{-}^{\nu} + R_{i\bar{j}k\bar{l}} \psi_{+}^i \psi_{+}^{\bar{j}} \psi_{-}^k \psi_{-}^{\bar{l}} \right)$$

which is amenable to rational curve counting. Specifically, the A model is defined by twisting worldsheet fermions into worldsheet scalars and vectors

as follows [21]:

$$\begin{aligned} \psi_+^i &\in \Gamma_{C^\infty}(\phi^*T^{1,0}X), & \psi_-^i &\in \Gamma_{C^\infty}(\overline{K}_\Sigma \otimes (\phi^*T^{0,1}X)^*), \\ \psi_+^{\bar{i}} &\in \Gamma_{C^\infty}(K_\Sigma \otimes (\phi^*T^{1,0}X)^*), & \psi_-^{\bar{i}} &\in \Gamma_{C^\infty}(\phi^*T^{0,1}X). \end{aligned}$$

The heterotic analogue of the A model, known as the A/2 model, is a twist of the (0, 2) NLSM

$$\begin{aligned} \frac{1}{\alpha'} \int_\Sigma d^2z \left((g_{\mu\nu} + iB_{\mu\nu}) \partial\phi^\mu \bar{\partial}\phi^\nu + \frac{i}{2} g_{\mu\nu} \psi_+^\mu D_{\bar{z}} \psi_+^\nu \right. \\ \left. + \frac{i}{2} h_{\alpha\beta} \lambda_-^\alpha D_z \lambda_-^\beta + F_{i\bar{j}a\bar{b}} \psi_+^i \psi_+^{\bar{j}} \lambda_-^a \lambda_-^{\bar{b}} \right) \end{aligned}$$

in which the fermions couple to bundles as follows:

$$\begin{aligned} \psi_+^i &\in \Gamma_{C^\infty}(\phi^*T^{1,0}X), & \lambda_-^a &\in \Gamma_{C^\infty}(\overline{K}_\Sigma \otimes \phi^*\mathcal{E}^*), \\ \psi_+^{\bar{i}} &\in \Gamma_{C^\infty}(K_\Sigma \otimes (\phi^*T^{1,0}X)^*), & \lambda_-^{\bar{a}} &\in \Gamma_{C^\infty}(\phi^*\mathcal{E}), \end{aligned}$$

where \mathcal{E} is a holomorphic vector bundle on X . Anomaly cancellation requires

$$\Lambda^{\text{top}}\mathcal{E}^* \cong K_X, \quad \text{ch}_2(\mathcal{E}) = \text{ch}_2(TX).$$

(The second statement is the Green–Schwarz anomaly cancellation condition generic to all heterotic theories; the first is a condition specific to the A/2 twist, an analogue of the condition that the closed string B model can only propagate on spaces X such that $K_X^{\otimes 2}$ is trivial [6, 21].) In fact, a specific choice of isomorphism $\Lambda^{\text{top}}\mathcal{E}^* \cong K_X$ is part of the data needed to define the path integral. Although both left- and right-movers have been twisted, the theory defined by the twisting above is not a topological field theory, since the worldsheet does not have supersymmetry on left-movers. Nevertheless it is sufficiently close to a true topological field theory to enable mathematical computations.

The Ramond–Ramond (RR) states of the A/2 model generalizing the A model states are counted by sheaf cohomology $H^q(X, \Lambda^p\mathcal{E}^\vee)$.

In general terms, we understand correlation functions in the A/2 model as follows (see [1] for a more complete discussion). For a space X with

holomorphic vector bundle $\mathcal{E} \rightarrow X$ satisfying

$$\det \mathcal{E}^* \cong K_X, \quad \text{ch}_2(\mathcal{E}) = \text{ch}_2(TX),$$

the classical contribution to a correlation function is

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle = \int_X \omega_1 \wedge \dots \wedge \omega_n,$$

where each ω_i is an element of $H^*(X, \Lambda^* \mathcal{E}^*)$, and corresponds to an operator \mathcal{O}_i . The correlation function can only be nonzero if

$$\omega_1 \wedge \dots \wedge \omega_n \in H^{\text{top}}(X, \Lambda^{\text{top}} \mathcal{E}^*)$$

and we get a number from this because of the isomorphism

$$\det \mathcal{E}^* \cong K_X$$

and the fact that $H^{\text{top}}(X, K_X) \cong \mathbf{C}$.

In sectors of nonzero instanton degree, each \mathcal{O}_i induces an element of $H^*(\mathcal{M}, \Lambda^* \mathcal{F}^*)$, where \mathcal{M} is the moduli space and \mathcal{F} a sheaf on \mathcal{M} induced by \mathcal{E} , as described in [1]. For example, if the moduli space \mathcal{M} admitted a universal instanton α , then $\mathcal{F} = R^0 \pi_* \alpha^* \mathcal{E}$. Schematically, if there are no $\psi^{\bar{i}}$, λ^a_- zero modes, then the contribution to a correlation function in a sector of nonzero instanton degree will be of the form

$$\int_{\mathcal{M}} \tilde{\omega}_1 \wedge \dots \wedge \tilde{\omega}_n,$$

where each $\tilde{\omega}_i$ is an element of $H^*(\mathcal{M}, \Lambda^* \mathcal{F}^*)$, and corresponds to an operator \mathcal{O}_i . In close analogy with the classical case, this contribution will be nonzero if

$$\tilde{\omega}_1 \wedge \dots \wedge \tilde{\omega}_n \in H^{\text{top}}(\mathcal{M}, \Lambda^{\text{top}} \mathcal{F}^*)$$

and we get a number from this because the conditions

$$\det \mathcal{E}^* \cong K_X, \quad \text{ch}_2(\mathcal{E}) = \text{ch}_2(TX)$$

imply (via Grothendieck–Riemann–Roch) that

$$\det \mathcal{F}^* \cong K_{\mathcal{M}}.$$

If there are $\psi_+^{\bar{i}}$, λ_-^a zero modes, then we have to make use of the four-Fermi terms, as described in [1]. Define

$$\mathcal{F}_1 \equiv R^1\pi_*\alpha^*\mathcal{E}, \quad \text{Obs} \equiv R^1\pi_*\alpha^*TX,$$

then one can formally identify the contribution of each four-Fermi term with an insertion of

$$H^1(\mathcal{M}, \mathcal{F}^* \otimes \mathcal{F}_1 \otimes (\text{Obs})^*).$$

Assuming equal numbers of $\psi_+^{\bar{i}}$, λ_-^a zero modes, correlation functions in such a sector will have the form

$$\int_{\mathcal{M}} \tilde{\omega}_1 \wedge \cdots \wedge \tilde{\omega}_n \wedge \alpha,$$

where the $\tilde{\omega}_i$ are as before and α is a wedge product of cohomology classes associated with four-Fermi terms. Altogether the contribution can only be nonzero if

$$\tilde{\omega}_1 \wedge \cdots \wedge \tilde{\omega}_n \wedge \alpha \in H^{\text{top}}(\mathcal{M}, \Lambda^{\text{top}}\mathcal{F}^* \otimes \Lambda^{\text{top}}\mathcal{F}_1 \otimes \Lambda^{\text{top}}\text{Obs}^*)$$

and we get a number from this because in these circumstances the conditions

$$\det \mathcal{E}^* \cong K_X, \quad \text{ch}_2(\mathcal{E}) = \text{ch}_2(TX)$$

imply (via Grothendieck–Riemann–Roch) that

$$\det \mathcal{F}^* \otimes \det \mathcal{F}_1 \otimes \det \text{Obs}^* \cong K_{\mathcal{M}}.$$

Now, let us begin to specialize to examples of the form we shall discuss in this paper. Consider a projective toric variety $X = X_{\Sigma}$ over \mathbf{C} of dimension n with fan Σ . The tangent bundle TX is defined by a cokernel of the form

$$0 \longrightarrow \mathcal{O}^{\oplus r} \xrightarrow{E} \bigoplus_{i=1}^n \mathcal{O}(\vec{q}_i) \longrightarrow TX \longrightarrow 0,$$

where r is the rank of the Picard lattice, whose complexification we denote as

$$W = \text{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{C}.$$

We will often denote $\mathcal{O}_X^{\oplus r}$ by $W \otimes \mathcal{O}_X$. The map E acts by mapping the a th \mathcal{O} as

$$\mathcal{O} \xrightarrow{q_{ai}\phi_i} \mathcal{O}(\vec{q}_i),$$

where $\phi_i \in \Gamma(\mathcal{O}(\vec{q}_i))$ is a homogeneous coordinate on the toric variety (see for example [22]).

Now, we shall consider deformations \mathcal{E} of the tangent bundle above, defined by cokernels

$$0 \longrightarrow \mathcal{O}^{\oplus r} \xrightarrow{E} \bigoplus_{i=1}^n \mathcal{O}(\vec{q}_i) \longrightarrow \mathcal{E} \longrightarrow 0$$

for more general maps E . Each element of E will be a polynomial. We will distinguish two types of contributions to E : “linear” and “nonlinear” deformations. Linear deformations involve monomials containing a single homogeneous coordinate (as in all of the maps defining the tangent bundle). Nonlinear deformations involve monomials containing a product of more than one homogeneous coordinate.

We will use the linear sigma model (LSM) moduli space \mathcal{M} . As explained in, e.g., [23], for the case above, this is constructed by expanding each of the homogeneous coordinates on X in a basis of zero modes on \mathbf{P}^1 , and interpreting the coefficients in the expansion as homogeneous coordinates on the moduli space. If

$$X = \mathbf{C}^n // (\mathbf{C}^\times)^r,$$

where $(\mathbf{C}^\times)^r$ acts on \mathbf{C}^n with weights \vec{q}_i , then the LSM moduli space of maps of degree \vec{d} is

$$\mathcal{M} = \left(\bigoplus_{i=1}^n H^0(\mathbf{P}^1, \mathcal{O}(\vec{q}_i \vec{d})) \right) // (\mathbf{C}^\times)^r.$$

It can be shown that the LSM moduli space \mathcal{M} is smooth whenever the original toric variety is. (The basic point is that if we describe the toric variety as $(\mathbf{C}^n - E)/G$, then singularities are at fixed points of G . See [20, Section 4.1] for further details.)

The induced sheaves $\mathcal{F}, \mathcal{F}_1$ can be constructed in an analogous fashion [1], by expanding worldsheet GLSM fermions in a basis of zero modes and interpreting the coefficients as line bundles over the moduli space. Specifically,

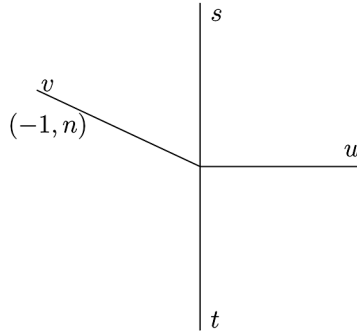


Figure 1: The fan for \mathbf{F}_n .

following the methods of [1], one finds for present case that

$$0 \longrightarrow \mathcal{O}^{\oplus r} \xrightarrow{E'^T} \bigoplus_{i=1}^n H^0(\mathbf{P}^1, \mathcal{O}(\vec{q}_i \vec{d})) \otimes_{\mathbf{C}} \mathcal{O}(\vec{q}_i) \longrightarrow \mathcal{F} \longrightarrow 0,$$

$$\mathcal{F}_1 \cong \bigoplus_{i=1}^n H^1(\mathbf{P}^1, \mathcal{O}(\vec{q}_i \vec{d})) \otimes_{\mathbf{C}} \mathcal{O}(\vec{q}_i).$$

The map E' in the definition of \mathcal{F} is induced from the corresponding map in the definition of \mathcal{E} . It is constructed by taking the map E in \mathcal{E} (a polynomial in homogeneous coordinates) and expanding in terms of homogeneous coordinates on the worldsheet \mathbf{P}^1 . The components of the induced map E' are then the coefficients of various monomials in the homogeneous coordinates on \mathbf{P}^1 .

To explain how the map is induced in more detail, let us consider the example of a Hirzebruch surface \mathbf{F}_n . To set notation, describe the Hirzebruch surface by the toric fan in figure 1.

From the fan, we read off the relations between toric divisors

$$D_u = D_v, \quad D_t = D_s + nD_v$$

and the Stanley–Reisner ideal

$$D_u D_v = 0 = D_s D_t.$$

The homogeneous coordinates u, v, s, t (corresponding to the four toric divisors) have the following weights under two \mathbf{C}^\times actions:

$$\begin{array}{cccc} u & v & s & t \\ \hline 1 & 1 & 0 & n \\ 0 & 0 & 1 & 1. \end{array}$$

We describe a deformation \mathcal{E}^* of the cotangent bundle as the cokernel

$$0 \longrightarrow \mathcal{E}^* \longrightarrow \mathcal{O}(-1, 0)^{\oplus 2} \oplus \mathcal{O}(0, -1) \oplus \mathcal{O}(-n, -1) \xrightarrow{E} W \otimes \mathcal{O} \longrightarrow 0,$$

where W is a two-dimensional vector space

$$E = \begin{bmatrix} Ax & Bx \\ \gamma_1 s & \gamma_2 s \\ \alpha_1 t + s f_1(u, v) & \alpha_2 t + s f_2(u, v) \end{bmatrix} \tag{1}$$

with

$$x \equiv \begin{bmatrix} u \\ v \end{bmatrix}$$

A, B constant 2×2 matrices, $\gamma_1, \gamma_2, \alpha_1, \alpha_2$ constants, and $f_{1,2}(u, v)$ homogeneous polynomials of degree n . (The matrices A, B and $\gamma_{1,2}$ define linear deformations of the tangent bundle; the functions $s f_{1,2}(u, v)$ define nonlinear deformations.)

To demonstrate the technology, consider for a moment maps of degree $(1, 0)$. In this case, we get the induced sheaf

$$0 \longrightarrow \mathcal{F}^* \longrightarrow \mathcal{O}(-1, 0)^4 \oplus \mathcal{O}(0, -1) \oplus \mathcal{O}(-n, -1)^{n+1} \xrightarrow{E'} W \otimes \mathcal{O} \longrightarrow 0,$$

where the map E' is induced from the map E by expanding fields in zero modes and picking off terms with the same homogeneous coordinates on \mathbf{P}^1 . Let us work through that in detail to illustrate the result. In the degree $(1, 0)$ sector, we expand

$$\begin{aligned} u &= u_0 a + u_1 b, \\ v &= v_0 a + v_1 b, \\ s &= s_0, \\ t &= t_0 a^n + t_1 a^{n-1} b + \dots + t_n b^n, \end{aligned}$$

where a, b are homogeneous coordinates on \mathbf{P}^1 . Then, in the original map E , we replace each field u, v, s, t by its expansion in zero modes above, and

pick off terms with the same homogeneous coordinates. In this fashion, we find

$$E' = \begin{bmatrix} \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} x' & \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} x' \\ \gamma_1 s_0 & \gamma_2 s_0 \\ \alpha_1 t_0 + s_0 f_{10} u_0^n + s_0 f_{11} u_0^{n-1} v_0 & \alpha_2 t_0 + s_0 f_{20} u_0^n + s_0 f_{21} u_0^{n-1} v_0 \\ + \cdots + s_0 f_{1n} v_0^n & + \cdots + s_0 f_{2n} v_0^n \\ \alpha_1 t_1 + s_0 f_{10}(n u_0^{n-1} u_1) + s_0 f_{11} u_0^{n-1} v_1 & \alpha_2 t_1 + s_0 f_{20}(n u_0^{n-1} u_1) + s_0 f_{21} u_0^{n-1} v_1 \\ + (n-1) s_0 f_{11} u_0^{n-2} u_1 v_0 + \cdots & + (n-1) s_0 f_{21} u_0^{n-2} u_1 v_0 + \cdots \\ \dots & \dots \end{bmatrix}, \tag{2}$$

where

$$x' = [u_0, v_0, u_1, v_1]^T$$

and

$$f_i(u, v) = f_{i0} u^n + f_{i1} u^{n-1} v + \cdots + f_{in} v^n.$$

In E' , the lines with t_0 , for example, correspond to coefficients of a^n , the lines with t^1 correspond to coefficients of $a^{n-1}b$, and so forth.

It can be shown in general that \mathcal{F} is locally free whenever \mathcal{E} is locally free [20]. Briefly, \mathcal{F} will be locally free whenever E' is surjective. At any point on the GLSM moduli space, pick a point on \mathbf{P}^1 at which the corresponding map is nondegenerate, then the image of E' is the image of E , hence surjectivity of E implies surjectivity of E' .

3 Example: projective space

Let us begin with an extremely simple example, namely \mathbf{P}^n . We will consider what appears formally to be a deformation of the tangent bundle of \mathbf{P}^n , defined by \mathcal{E} below

$$0 \longrightarrow \mathcal{E}^* \longrightarrow Z_0 \xrightarrow{E} W \otimes \mathcal{O} \longrightarrow 0,$$

where

$$Z_0 = \mathcal{O}(-1)^{\oplus n+1}, \quad E = Ax,$$

where W is a one-dimensional vector space, x is a vector of homogeneous coordinates on \mathbf{P}^n and A a constant $(n + 1) \times (n + 1)$ matrix. We say this appears to be a deformation; however, the tangent bundle of \mathbf{P}^n admits no deformations, hence the matrix A encodes, for nondegenerate A , mere reparametrizations. By contrast, for $\mathbf{P}^1 \times \mathbf{P}^1$, which we shall study in the next section, generic deformations of the tangent bundle yield bundles which are not isomorphic to the original tangent bundle.

Since we are simply giving a more complicated description of \mathbf{P}^n with its tangent bundle, the quantum sheaf cohomology ring we compute should exactly match the ordinary quantum cohomology ring, which is what we shall find. This example will serve as a useful computational exercise, but we will not start generating new results until the next section.

First, let us consider the classical cohomology ring. A nonzero correlation function arises from correlators of total degree n , equal to the dimension of \mathbf{P}^n . Classical correlation functions are then a map

$$\text{Sym}^n W = H^0(\text{Sym}^n W \otimes \mathcal{O}) \longrightarrow H^n(\Lambda^n \mathcal{E}^*).$$

To determine the map, we use the generalized Koszul complex associated to $\Lambda^n \mathcal{E}^*$

$$0 \longrightarrow \Lambda^n \mathcal{E}^* \longrightarrow \Lambda^n Z_0 \longrightarrow \Lambda^{n-1} Z_0 \otimes W \longrightarrow \dots \longrightarrow \text{Sym}^n W \otimes \mathcal{O} \longrightarrow 0$$

which factorizes into a series of maps

$$0 \longrightarrow \Lambda^n \mathcal{E}^* \longrightarrow \Lambda^n Z_0 \longrightarrow S_{n-1} \longrightarrow 0, \tag{3}$$

$$0 \longrightarrow S_i \longrightarrow \Lambda^i Z_0 \otimes \text{Sym}^{n-i} W \longrightarrow S_{i-1} \longrightarrow 0, \tag{4}$$

$$0 \longrightarrow S_1 \longrightarrow Z \otimes \text{Sym}^{n-1} W \longrightarrow \text{Sym}^n W \otimes \mathcal{O} \longrightarrow 0. \tag{5}$$

Now, $H^j(\Lambda^i Z_0)$ will vanish unless $j = n$, $i = n + 1$ (or $i = j = 0$, but we shall suppress that case as it will not be pertinent for our computations). Thus, from (3) we find

$$H^n(\Lambda^n \mathcal{E}^*) \xrightarrow{\sim} H^{n-1}(S_{n-1})$$

from (4) we find

$$H^{i-1}(S_{i-1}) \xrightarrow{\sim} H^i(S_i)$$

for $2 \leq i \leq n - 1$, and from (5) we find

$$H^0(\text{Sym}^n W \otimes \mathcal{O}) \xrightarrow{\sim} H^1(S_1)$$

which implies that the map

$$\text{Sym}^n W \xrightarrow{\sim} H^n(\Lambda^n \mathcal{E}^*)$$

is an isomorphism, as indicated. (As a consistency check, note that since W is 1D, $\text{Sym}^n W$ is also 1D.)

Now, in principle, the factorization of the generalized Koszul complex will stop giving isomorphisms if ever we need to compute $H^n(\Lambda^{n+1} Z_0)$. This will happen if we consider correlation functions with correlators of degree greater than n . For example, if we have degree $n + 1$ correlators, then the correlation function computes a map

$$\text{Sym}^{n+1} W \longrightarrow H^{n+1}(\Lambda^{n+1} \mathcal{E}^*) = 0.$$

In this (trivial) case, we have the generalized Koszul complex

$$\begin{aligned} 0 &\longrightarrow \Lambda^{n+1} \mathcal{E}^*(=0) \longrightarrow \Lambda^{n+1} Z_0 \longrightarrow \Lambda^n Z_0 \otimes W \\ &\longrightarrow \dots \longrightarrow \text{Sym}^{n+1} W \otimes \mathcal{O} \longrightarrow 0 \end{aligned}$$

which factorizes as

$$0 \longrightarrow \Lambda^{n+1} \mathcal{E}^*(=0) \longrightarrow \Lambda^{n+1} Z_0 \longrightarrow S_n \longrightarrow 0, \tag{6}$$

$$0 \longrightarrow S_i \longrightarrow \Lambda^i Z_0 \otimes \text{Sym}^{n+1-i} W \longrightarrow S_{i-1} \longrightarrow 0, \tag{7}$$

$$0 \longrightarrow S_1 \longrightarrow Z \otimes \text{Sym}^n W \longrightarrow \text{Sym}^{n+1} W \otimes \mathcal{O} \longrightarrow 0. \tag{8}$$

As before, from (8) we have

$$H^0(\text{Sym}^{n+1} W \otimes \mathcal{O}) \xrightarrow{\sim} H^1(S_1)$$

and from (7) we have

$$H^{i-1}(S_{i-1}) \xrightarrow{\sim} H^i(S_i)$$

for $2 \leq i \leq n$. Finally from (6) we have

$$H^n(S_n) \xrightarrow{\sim} H^n(\Lambda^{n+1} Z_0).$$

Thus, the original correlation function necessarily vanishes

$$\text{Sym}^{n+1} W \longrightarrow H^{n+1}(\Lambda^{n+1} \mathcal{E}^*) = 0$$

but

$$\text{Sym}^{n+1} W \xrightarrow{\sim} H^n(\Lambda^{n+1} Z_0).$$

From [20, Section 3.3], the group $H^n(\Lambda^{n+1} Z_0)$ is a 1D vector space generated by

$$\det(A\psi) = (\det A) \psi^{n+1},$$

where ψ is a basis element for W .

Thus, we find the classical sheaf cohomology ring is of the form

$$\mathbf{C}[\psi]/(\det(A\psi)) \cong \mathbf{C}[\psi]/(\psi^{n+1}).$$

Now, let us turn to the quantum sheaf cohomology ring. We shall compute this by first computing the (classical) sheaf cohomology ring in any sector of fixed instanton degree, then relating sectors of different instanton number.

In a sector of instanton number d , the LSM moduli space of \mathbf{P}^n is easily computed to be $\mathbf{P}^{(n+1)(d+1)-1}$. The induced bundle over the LSM moduli space is \mathcal{F} , where

$$0 \longrightarrow \mathcal{F}^* \longrightarrow Z \xrightarrow{E'} W \otimes \mathcal{O} \longrightarrow 0,$$

where

$$Z = \mathcal{O}(-1)^{\oplus(n+1)(d+1)},$$

W is the same one-dimensional vector space from previously, and

$$E' = \begin{bmatrix} Ax_0 \\ Ax_1 \\ \vdots \\ Ax_d \end{bmatrix},$$

where x_i is a $(n + 1)$ -element vector of coefficients of fixed degree in the expansion of homogeneous coordinates of \mathbf{P}^n in zero modes.

We can now re-use the classical results above. For fixed instanton degree d , the sheaf cohomology ring is

$$\mathbf{C}[\psi]/\left((\det(A\psi))^{d+1}\right) \cong \mathbf{C}[\psi]/\left(\psi^{(n+1)(d+1)}\right).$$

Therefore, to preserve kernels, any relation between correlation functions in different sectors of fixed degree must be generated by

$$\langle \mathcal{O} \rangle_0 \propto \langle \mathcal{O}(\det(A\psi))^d \rangle_d.$$

(This ensures that if \mathcal{O} is an element of the quotiented ideal in the zero-degree sector, so that $\langle \mathcal{O} \rangle_0$ vanishes, then its image $\mathcal{O}(\det(A\psi))^d$ will be an element of the quotiented ideal in the sector of degree d , so that the corresponding correlation function also vanishes.)

The relation above then implies that

$$\det(A\psi) = q$$

for some constant q . (For example, this follows immediately in $d = 1$, then higher degrees must just be a power.) Since $\det(A\psi) = (\det A)\psi$, this is equivalent to the relation

$$\psi^{n+1} = q'$$

which is the standard quantum cohomology relation for a projective space \mathbf{P}^n .

In the next sections, our tangent bundle deformations will, in general, yield bundles that are not isomorphic to the tangent bundle, so the quantum sheaf cohomology relations will be nontrivial.

4 Example: product of projective spaces

Mathematical computations of quantum sheaf cohomology have previously [1, 3, 5, 18] relied on brute-force Čech cohomology representations. One of the advancements of this paper and [20] is the use of purely analytic methods to derive quantum sheaf cohomology.

We will illustrate these advances through an explicit computation for general deformations of the tangent bundle of $\mathbf{P}^1 \times \mathbf{P}^1$. In particular, previously special deformations of the tangent bundle of $\mathbf{P}^1 \times \mathbf{P}^1$ have been computed with brute-force Čech techniques, so this seems an appropriate

example to generalize here. We will begin by examining classical cup products for $\mathbf{P}^1 \times \mathbf{P}^1$, then classical cup products for $\mathbf{P}^n \times \mathbf{P}^m$, and then we will describe the quantum sheaf cohomology ring for $\mathbf{P}^1 \times \mathbf{P}^1$, which will ultimately be determined by the classical computations on products of more general projective spaces.

4.1 Classical cup products on $\mathbf{P}^1 \times \mathbf{P}^1$

In this section we will discuss how to compute classical cup products in the sheaf cohomology, without having to work through a Čech cohomology computation.

Define $V = \Gamma(\mathcal{O}(1, 0))$, $\tilde{V} = \Gamma(\mathcal{O}(0, 1))$, $W = \mathbf{C}^2$. Define

$$Z_0 \equiv (V \otimes \mathcal{O}(-1, 0)) \oplus (\tilde{V} \otimes \mathcal{O}(0, -1)).$$

Then, the cotangent bundle deformation \mathcal{E}^* is the kernel

$$0 \longrightarrow \mathcal{E}^* \longrightarrow Z_0 \xrightarrow{E} W \otimes \mathcal{O} \longrightarrow 0,$$

where

$$E = \begin{bmatrix} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{bmatrix},$$

where x, \tilde{x} are vectors of homogeneous coordinates on each \mathbf{P}^1 factor.

First, let us compute the classical sheaf cohomology ring. Classical correlation functions are a map¹

$$\text{Sym}^2 W = H^0(\text{Sym}^2 W \otimes \mathcal{O}) \longrightarrow H^2(\Lambda^2 \mathcal{E}^*)$$

¹More formally, we could think of classical correlation functions and the map above as an element of

$$\text{Ext}^2(\text{Sym}^2(W \otimes \mathcal{O}), \Lambda^2 \mathcal{E}^*)$$

which corresponds to the exact sequence (9). Breaking that long sequence into two short exact sequences along Q corresponds to writing the Ext element above as a product of elements of

$$\text{Ext}^1(Q, \Lambda^2 \mathcal{E}^*), \quad \text{Ext}^1(\text{Sym}^2(W \otimes \mathcal{O}), Q)$$

which correspond to the short exact sequences (10) and (11).

and we will determine the ring structure by computing the kernel of that map. We will use the generalized Koszul complex of $\Lambda^2 \mathcal{E}^*$

$$0 \longrightarrow \Lambda^2 \mathcal{E}^* \longrightarrow \Lambda^2 Z_0 \longrightarrow Z_0 \otimes (W \otimes \mathcal{O}) \longrightarrow \text{Sym}^2 (W \otimes \mathcal{O}) \longrightarrow 0. \quad (9)$$

It remains to compute the cup product above. First, split the long exact sequence (9) into a pair of short exact sequences

$$0 \longrightarrow \Lambda^2 \mathcal{E}^* \longrightarrow \Lambda^2 Z_0 \longrightarrow Q \longrightarrow 0, \quad (10)$$

$$0 \longrightarrow Q \longrightarrow Z_0 \otimes (W \otimes \mathcal{O}) \longrightarrow \text{Sym}^2 (W \otimes \mathcal{O}) \longrightarrow 0 \quad (11)$$

which define Q .

Next, we shall evaluate the kernel of that map, that product, which will give us the classical sheaf cohomology ring structure.

The short exact sequence (11) induces a map

$$\delta_1 : H^0 (\text{Sym}^2 W \otimes \mathcal{O}) \longrightarrow H^1 (Q)$$

(from the associated long exact sequence). Moreover, because

$$H^* (Z_0 \otimes W) = 0$$

the map δ_1 above is an isomorphism.

The other short exact sequence, (10), induces

$$0 \longrightarrow H^1 (\Lambda^2 Z_0) \longrightarrow H^1(Q) \xrightarrow{\delta_2} H^2 (\Lambda^2 \mathcal{E}^*) \longrightarrow 0$$

using the fact that

$$H^1 (\Lambda^2 \mathcal{E}^* = K_{\mathbf{P}^1 \times \mathbf{P}^1}) = 0, \quad H^2 (\Lambda^2 Z_0) = 0.$$

The classical cup product is then the composition

$$H^0 (\text{Sym}^2 W \otimes \mathcal{O}) \xrightarrow{\delta_1} H^1(Q) \xrightarrow{\delta_2} H^2 (\Lambda^2 \mathcal{E}^*). \quad (12)$$

We have seen that δ_1 is an isomorphism, but δ_2 has a nontrivial kernel. Specifically, since

$$\Lambda^2 Z_0 = (\Lambda^2 V \otimes \mathcal{O}(-2, 0)) \oplus (\Lambda^2 \tilde{V} \otimes \mathcal{O}(0, -2)) \oplus (V \otimes \tilde{V} \otimes \mathcal{O}(-1, -1))$$

we see that the kernel of the classical cup product is one-dimensional

$$H^1(\Lambda^2 Z_0) = \Lambda^2 V \oplus \Lambda^2 \tilde{V}.$$

In fact, it can be shown [20, Section 3.3] that the kernel of the cup product map (12) is defined by the relations

$$\det(\psi A + \tilde{\psi} B) = 0, \tag{13}$$

$$\det(\psi C + \tilde{\psi} D) = 0. \tag{14}$$

These are the classical sheaf cohomology ring relations.

Let us check that this correctly reproduces the results of [3]. In that paper, $A = D = I$

$$B = \begin{bmatrix} \epsilon_1 & \epsilon_2 \\ \epsilon_3 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & 0 \end{bmatrix}.$$

There, the classical cohomology ring is given by

$$\begin{aligned} \psi^2 + \epsilon_1 \psi \tilde{\psi} - \epsilon_2 \epsilon_3 \tilde{\psi}^2 &= 0, \\ \tilde{\psi}^2 + \gamma_1 \psi \tilde{\psi} - \gamma_2 \gamma_3 \psi^2 &= 0. \end{aligned}$$

Applying the general methods above to the matrices A, B, C, D here, we find that

$$\begin{aligned} \det(\psi A + \tilde{\psi} B) &= \psi^2 + \epsilon_1 \psi \tilde{\psi} - \epsilon_2 \epsilon_3 \tilde{\psi}^2, \\ \det(\psi C + \tilde{\psi} D) &= \tilde{\psi}^2 + \gamma_1 \psi \tilde{\psi} - \gamma_2 \gamma_3 \psi^2 \end{aligned}$$

and so we recover the results of [3] for the classical cohomology ring as a special case. Similarly, it is straightforward to check that this also agrees with the general results of [4], as we shall review later in Section 6.

4.2 Classical cup products on $\mathbf{P}^n \times \mathbf{P}^m$

Let us now quickly repeat the analysis of the previous subsection for a more general product of projective spaces, $\mathbf{P}^n \times \mathbf{P}^m$. In the next section, we will compute the quantum sheaf cohomology ring for $\mathbf{P}^1 \times \mathbf{P}^1$, which will be determined by classical computations on $\mathbf{P}^n \times \mathbf{P}^m$.

As before, define $V = \Gamma(\mathcal{O}(1, 0))$, $\tilde{V} = \Gamma(\mathcal{O}(0, 1))$, $W = \mathbf{C}^2$. Define

$$Z = (V \otimes \mathcal{O}(-1, 0)) \oplus (\tilde{V} \otimes \mathcal{O}(0, -1)).$$

Then, as before, the cotangent bundle deformation \mathcal{E}^* is the kernel

$$0 \longrightarrow \mathcal{E}^* \longrightarrow Z \xrightarrow{E} W \otimes \mathcal{O} \longrightarrow 0,$$

where

$$E = \begin{bmatrix} \tilde{A}x & \tilde{B}x \\ \tilde{C}\tilde{x} & \tilde{D}\tilde{x} \end{bmatrix},$$

where x, \tilde{x} are vectors of homogeneous coordinates on $\mathbf{P}^n, \mathbf{P}^m$, respectively, \tilde{A}, \tilde{B} are $(n + 1) \times (n + 1)$ matrices and \tilde{C}, \tilde{D} are $(m + 1) \times (m + 1)$ matrices.

As before, we think of classical correlation functions in this theory as maps

$$\text{Sym}^{n+m}W \longrightarrow H^{n+m}(\Lambda^{\text{top}}\mathcal{E}^*)$$

and we compute the kernel, using the generalized Koszul complex associated to $\Lambda^{\text{top}}\mathcal{E}^*$

$$\begin{aligned} 0 \longrightarrow \Lambda^{n+m}\mathcal{E}^* &\longrightarrow \Lambda^{n+m}Z \longrightarrow \Lambda^{n+m-1}Z \otimes W \\ &\longrightarrow \dots \longrightarrow \text{Sym}^{n+m}W \otimes \mathcal{O} \longrightarrow 0. \end{aligned}$$

To do computations, we split this into short exact sequences

$$0 \longrightarrow \Lambda^{n+m}\mathcal{E}^* \longrightarrow \Lambda^{n+m}Z \longrightarrow S_{n+m-1} \longrightarrow 0, \tag{15}$$

$$0 \longrightarrow S_i \longrightarrow \Lambda^i Z \otimes \text{Sym}^{n+m-i}W \longrightarrow S_{i-1} \longrightarrow 0, \tag{16}$$

$$0 \longrightarrow S_1 \longrightarrow Z \otimes \text{Sym}^{n+m-1}W \longrightarrow \text{Sym}^{n+m}W \otimes \mathcal{O} \longrightarrow 0. \tag{17}$$

Now, $H^j(\Lambda^i Z)$ will vanish unless $j = i - 1 = n, m$ (see for example [20]) (or, alternatively, if $i = j = 0$, but we shall suppress that case as it will not

be pertinent for our computations). Thus, from (15), we find

$$H^{n+m-1}(S_{n+m-1}) \xrightarrow{\sim} H^{n+m}(\Lambda^{n+m}\mathcal{E}^*)$$

from (17) we find

$$H^0(\text{Sym}^{n+m}W \otimes \mathcal{O}) \xrightarrow{\sim} H^1(S_1)$$

and from (16) we find a surjective map

$$H^{i-1}(S_{i-1}) \longrightarrow H^i(S_i)$$

for $2 \leq i \leq n + m - 1$. If $i - 1 \neq n, m$, then the surjective map above is an isomorphism. If $i - 1$ is either n or m , then it has a nontrivial kernel, given by $H^{i-1}(\Lambda^i Z \otimes \text{Sym}^{n+m-i}W)$.

If $i - 1 = n \neq m$, then $H^{i-1}(\Lambda^i Z) \cong \Lambda^{\text{top}}V$, and it can be shown [20, Section 3.3] that this is generated by

$$\det(\psi\tilde{A} + \tilde{\psi}\tilde{B}),$$

where $\{\psi, \tilde{\psi}\}$ is a basis for W . Thus, the kernel of $H^n(S_n) \rightarrow H^{n+1}(S_{n+1})$ is generated by

$$\det(\psi\tilde{A} + \tilde{\psi}\tilde{B}).$$

The case $i - 1 = m \neq n$ is nearly identical, so we omit its description. If $i - 1 = n = m$, the result is very similar. In this case

$$H^n(\Lambda^{n+1}Z) = \Lambda^{n+1}V \oplus \Lambda^{n+1}\tilde{V}$$

and the kernel of $H^n(S_n) \rightarrow H^{n+1}(S_{n+1})$ is generated by

$$\det(\psi\tilde{A} + \tilde{\psi}\tilde{B}), \quad \det(\psi\tilde{C} + \tilde{\psi}\tilde{D}).$$

Putting this together, we find that the classical sheaf cohomology ring of $\mathbf{P}^n \times \mathbf{P}^m$ with bundle \mathcal{E} is generated by $\psi, \tilde{\psi}$ with relations

$$\det(\psi\tilde{A} + \tilde{\psi}\tilde{B}) = 0 = \det(\psi\tilde{C} + \tilde{\psi}\tilde{D}).$$

4.3 Quantum sheaf cohomology ring on $\mathbf{P}^1 \times \mathbf{P}^1$

Define

$$Q \equiv \det(\psi A + \tilde{\psi} B),$$

$$\tilde{Q} \equiv \det(\psi C + \tilde{\psi} D).$$

In this section, we will show that the quantum sheaf cohomology ring of $\mathbf{P}^1 \times \mathbf{P}^1$, with bundle \mathcal{E} defined earlier, is given by

$$\mathbf{C}[\psi, \tilde{\psi}] / (Q - q, \tilde{Q} - \tilde{q}).$$

First, we shall derive the form of the cohomology ring in each fixed instanton sector, then, we shall find relations between the sectors.

We shall begin by deriving the ring for fixed instanton degree (d, e) . As outlined earlier, the LSM moduli space is computed to be

$$\mathcal{M} = \mathbf{P}^{2d+1} \times \mathbf{P}^{2e+1}.$$

Define Z to be the following sheaf on \mathcal{M} :

$$Z \equiv \left(\text{Sym}^d U \otimes V \otimes \mathcal{O}(-1, 0) \right) \oplus \left(\text{Sym}^e U \otimes \tilde{V} \otimes \mathcal{O}(0, -1) \right).$$

The induced sheaf \mathcal{F}^* is the kernel

$$0 \longrightarrow \mathcal{F}^* \longrightarrow Z \longrightarrow W \otimes \mathcal{O}_{\mathcal{M}} \longrightarrow 0,$$

where

$$U = \Gamma(\mathbf{P}^1, \mathcal{O}(1)), \quad V = \Gamma(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(1, 0)),$$

$$\tilde{V} = \Gamma(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(0, 1)), \quad W = \mathbf{C}^2$$

which is naturally induced from the short exact sequence defining \mathcal{E}^* , as discussed in Section 2.

The desired correlation function in sector (d, e) can be computed as a classical sheaf cohomology cup product on $\mathcal{M} = \mathbf{P}^{2d+1} \times \mathbf{P}^{2e+1}$. As we have already computed classical sheaf cohomology on a product of projective

spaces, we can apply our results from the previous subsection. The induced maps are such that, for example

$$\tilde{A} = \text{diag}(A, A, \dots, A)$$

($d + 1$ copies), hence the classical sheaf cohomology ring relations are

$$\begin{aligned} \det(\psi\tilde{A} + \tilde{\psi}\tilde{B}) &= \det(\psi A + \tilde{\psi}B)^{d+1} = 0, \\ \det(\psi\tilde{C} + \tilde{\psi}\tilde{D}) &= \det(\psi C + \tilde{\psi}D)^{e+1} = 0 \end{aligned}$$

and so we immediately find that for fixed degree (d, e) , the sheaf cohomology groups

$$H^*(\mathcal{M}, \Lambda^* \mathcal{F}^*)$$

live in the polynomial ring

$$\text{Sym } W / (Q^{d+1}, \tilde{Q}^{e+1}).$$

For example, for degree $(d, e) = (1, 0)$, the kernel is spanned by the four polynomials

$$Q^2, \tilde{Q}\psi^2, \tilde{Q}\psi\tilde{\psi}, \tilde{Q}\tilde{\psi}^2$$

and it is straightforward to check that this is a correct property of the correlation functions give in, e.g., [1, equations (21) to (30)].

It remains to derive the operator product ring, the quantum sheaf cohomology ring utilizing the structure derived.

As there are no four-Fermi contributions ($\mathcal{F}_1 = \text{Obs} = 0$), we expect from existence of operator products that there should be relations between correlation functions in different instanton sectors, of the form

$$\langle \mathcal{O} \rangle_{d,e} \propto \langle \mathcal{O} R_{d,e,d',e'} \rangle_{d',e'} \tag{18}$$

for all \mathcal{O} and some fixed operator $R_{d,e,d',e'}$. For example

$$\langle \mathcal{O} \rangle_{0,0} \propto \langle \mathcal{O} Q \rangle_{1,0}$$

which suggests $Q = q$ for some proportionality constant q , and

$$\langle \mathcal{O} \rangle_{0,0} \propto \langle \mathcal{O} \tilde{Q} \rangle_{0,1}$$

which suggests $\tilde{Q} = \tilde{q}$ for some proportionality constant \tilde{q} . Equation (18) is merely the generalization to arbitrary instanton degrees. Because of compatibility with the kernels above (i.e., maps must send kernels to (subsets of) kernels, and must map top-forms to top-forms), the relations (18) should be of the form

$$\langle \mathcal{O} \rangle_{d,e} \propto \langle \mathcal{O} Q^{d'-d} \tilde{Q}^{e'-e} \rangle_{d',e'}$$

hence

$$\langle \mathcal{O} \rangle_{d,e} = A_{d,e,d',e'} \langle \mathcal{O} Q^{d'-d} \tilde{Q}^{e'-e} \rangle_{d',e'}$$

for some constant $A_{d,e,d',e'}$. We assume that the constant $A_{d,e,d',e'}$ has the form

$$A_{d,e,d',e'} = q^{d'-d} \tilde{q}^{e'-e}$$

for some constants q, \tilde{q} . Note that mathematically this is an assumption, not a derivation; we justify this assumption by the fact that this is the standard form of nonperturbative corrections to operator products, and so we recover standard physics results.

Thus

$$\langle \mathcal{O} \rangle_{d,e} = q^{d'-d} \tilde{q}^{e'-e} \langle \mathcal{O} Q^{d'-d} \tilde{Q}^{e'-e} \rangle_{d',e'}$$

and in particular

$$\langle \psi \tilde{\psi} Q^d \tilde{Q}^e \rangle_{d,e} = q^d \tilde{q}^e \langle \psi \tilde{\psi} \rangle_{0,0}$$

from which we derive the quantum sheaf cohomology relations

$$Q \sim q, \quad \tilde{Q} \sim \tilde{q}$$

so that the quantum sheaf cohomology ring is given by

$$\mathbf{C}[\psi, \tilde{\psi}] / (Q - q, \tilde{Q} - \tilde{q}).$$

This matches the prediction of [4], and also specializes to the results in [1, 3].

As a consistency check, let us quickly observe how bundle isomorphisms preserve the ring above. Let $R \in GL(W)$, $P_1 \in GL(V)$, $P_2 \in GL(\tilde{V})$. Under

the action of this $GL(2)^3$

$$E \mapsto \begin{bmatrix} P_1Ax & P_1Bx \\ P_2C\tilde{x} & P_2D\tilde{x} \end{bmatrix} R.$$

As R also acts on $\psi, \tilde{\psi}$, its action falls out of the ring relations, and we are left with

$$\det(A\psi + B\tilde{\psi}) \mapsto \det(P_1(A\psi + B\tilde{\psi})) = \det P_1 \det(A\psi + B\tilde{\psi}),$$

$$\det(C\psi + D\tilde{\psi}) \mapsto \det(P_2(C\psi + D\tilde{\psi})) = \det P_2 \det(C\psi + D\tilde{\psi})$$

and so we see that by absorbing $\det P_i$ into q, \tilde{q} , the ring is preserved.

5 Example: Hirzebruch surface

Next, we shall compute quantum sheaf cohomology for a deformation of the tangent bundle of the Hirzebruch surface \mathbf{F}_n . We will use the same notation as earlier in Section 2. As in that section, the homogeneous coordinates u, v, s, t (corresponding to the four toric divisors) have the following weights under two \mathbf{C}^\times actions:

$$\begin{array}{cccc} u & v & s & t \\ \hline 1 & 1 & 0 & n \\ 0 & 0 & 1 & 1. \end{array}$$

We describe a deformation \mathcal{E}^* of the cotangent bundle as the kernel

$$0 \longrightarrow \mathcal{E}^* \longrightarrow Z \xrightarrow{E} W \otimes \mathcal{O} \longrightarrow 0,$$

where

$$Z = \mathcal{O}(-1, 0)^{\oplus 2} \oplus \mathcal{O}(0, -1) \oplus \mathcal{O}(-n, -1),$$

W is a two-dimensional vector space

$$E = \begin{bmatrix} Ax & Bx \\ \gamma_1s & \gamma_2s \\ \alpha_1t + sf_1(u, v) & \alpha_2t + sf_2(u, v) \end{bmatrix}$$

with

$$x \equiv \begin{bmatrix} u \\ v \end{bmatrix},$$

A, B constant 2×2 matrices, $\gamma_1, \gamma_2, \alpha_1, \alpha_2$ constants and $f_{1,2}(u, v)$ homogeneous polynomials of degree n .

First, we shall outline the classical cohomology ring. As before, we use the generalized Koszul complex associated to $\Lambda^2 \mathcal{E}^*$, split it into two short exact sequences, and compute the kernel of the map $\text{Sym}^2 W \rightarrow H^2(\Lambda^2 \mathcal{E}^*)$. The kernel arises from $H^1(\Lambda^2 Z)$, which is 2D. It can be shown [20, Section 3.3] that the kernel is generated by

$$\det(\psi A + \tilde{\psi} B), (\psi \gamma_1 + \tilde{\psi} \gamma_2) (\psi \alpha_1 + \tilde{\psi} \alpha_2).$$

Because we will be encountering these polynomials often, we shall assign them names as follows:

$$\begin{aligned} Q_{K1} &= \det(\psi A + \tilde{\psi} B), \\ Q_s &= \psi \gamma_1 + \tilde{\psi} \gamma_2, \\ Q_t &= \psi \alpha_1 + \tilde{\psi} \alpha_2. \end{aligned}$$

(This nomenclature is used in the companion paper [20].) Thus, the kernel in the degree $\vec{d} = 0$ sector is generated by $Q_{K1}, Q_s Q_t$.

Next, consider the sector of instanton degree $\vec{d} = (1, 0)$. The LSM moduli space has homogeneous coordinates $u_{0,1}, v_{0,1}, s, t_{0,\dots,n}$, with weights

$u_{0,1}$	$v_{0,1}$	s	$t_{0,\dots,n}$
1	1	0	n
0	0	1	1

with exceptional set

$$\{u_0 = u_1 = v_0 = v_1 = 0, s = t_0 = t_1 = \dots = t_n = 0\}.$$

The induced bundle \mathcal{F} is given by

$$0 \longrightarrow \mathcal{F}^* \longrightarrow Z \xrightarrow{E'} W \otimes \mathcal{O} \longrightarrow 0,$$

where

$$Z = \mathcal{O}(-1, 0)^{\oplus 4} \oplus \mathcal{O}(0, -1) \oplus \mathcal{O}(-n, -1)^{\oplus n+1}$$

and the map E' , induced from E , is given by

$$E' = \begin{bmatrix} Ax_0 & Bx_0 \\ Ax_1 & Bx_1 \\ \gamma_1 s & \gamma_2 s \\ \alpha_1 t_0 + s f_1(u_0, v_0) & \alpha_2 t_0 + s f_2(u_0, v_0) \\ \alpha_1 t_1 + s \dots & \alpha_2 t_1 + s \dots \\ \dots & \dots \\ \alpha_1 t_n + s \dots & \alpha_2 t_n + s \dots \end{bmatrix},$$

where we are using \dots to abbreviate full zero mode expansions of f_1, f_2 , as described earlier in e.g., the analogous case of equation (2). We use $s \dots$ merely to denote a series of terms which have s as a common factor.

We want to compute the kernel of the map $\text{Sym}^{n+4}W \rightarrow H^{n+4}(\Lambda^{n+4}\mathcal{F}^*)$, which we do using the generalized Koszul complex associated to $\Lambda^{n+4}\mathcal{F}^*$. Following the usual pattern, and using the exceptional set described above (and the primitive collection it determines as in [20]), we find that the map $H^3(S_3) \rightarrow H^4(S_4)$ fails to be an isomorphism (because $H^3(\Lambda^4 Z)$ is nonzero) and $H^{n+1}(S_{n+1}) \rightarrow H^{n+2}(S_{n+2})$ fails to be an isomorphism (because $H^{n+1}(\Lambda^{n+2}\mathcal{F})$ is nonzero). The kernel arising from the first is generated by [20, Section 3.3]

$$Q_{K1}^2 = \left(\det \left(\psi A + \tilde{\psi} B \right) \right)^2$$

and the kernel arising from the second is generated by [20, Section 3.3]

$$Q_s Q_t^{n+1} = \left(\psi \gamma_1 + \tilde{\psi} \gamma_2 \right) \left(\psi \alpha_1 + \tilde{\psi} \alpha_2 \right)^{n+1}.$$

In terms of correlation functions, the result above implies that

$$\langle \mathcal{O} \rangle_{\vec{d}=0} \propto \langle \mathcal{O} Q_{K1} Q_t^n \rangle_{\vec{d}=(1,0)}$$

which suggests that the OPE ring has the (partial) form

$$Q_{K1} Q_t^n = q_1 \tag{19}$$

for some parameter q_1 .

Next, consider the degree $\vec{d} = (0, 1)$ sector. The LSM moduli space has homogeneous coordinates $u, v, s_{0,1}, t_{0,1}$ (where the s_i and t_i are the coefficients in the zero-mode expansion of s, t). These coordinates have weights

$$\begin{array}{cccccc} u & v & s_0 & s_1 & t_0 & t_1 \\ \hline 1 & 1 & 0 & 0 & n & n \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}$$

and the exceptional set is given by

$$\{u = v = 0, s_0 = s_1 = t_0 = t_1 = 0\}.$$

The induced bundle \mathcal{F} is now given by

$$0 \longrightarrow \mathcal{F}^* \longrightarrow Z \xrightarrow{E'} W \otimes \mathcal{O} \longrightarrow 0,$$

where

$$Z = \mathcal{O}(-1, 0)^{\oplus 2} \oplus \mathcal{O}(0, -1)^{\oplus 2} \oplus \mathcal{O}(-n, -1)^{\oplus 2}$$

and the map E' , induced from E , is given by

$$E' = \begin{bmatrix} Ax & Bx \\ \gamma_1 s_0 & \gamma_2 s_0 \\ \gamma_1 s_1 & \gamma_2 s_1 \\ \alpha_1 t_0 + s_0 f_1(u, v) & \alpha_2 t_0 + s_0 f_2(u, v) \\ \alpha_1 t_1 + s_1 f_1(u, v) & \alpha_2 t_1 + s_1 f_2(u, v) \end{bmatrix}.$$

As before, we want to compute the kernel of the map $\text{Sym}^4 W \rightarrow H^4(\Lambda^4 \mathcal{F}^*)$, which we do using the generalized Koszul complex associated to $\Lambda^4 \mathcal{F}^*$. Following the usual pattern, and using the exceptional collection described above, we find that the map $H^1(S_1) \rightarrow H^2(S_2)$ fails to be an isomorphism (because $H^1(\Lambda^2 Z)$ is nonzero) and $H^3(S_3) \rightarrow H^4(\Lambda^4 \mathcal{F}^*)$ fails to be an isomorphism (because $H^3(\Lambda^4 Z)$ is nonzero). The kernel arising from the first is generated by

$$Q_{K1} = \det \left(\psi A + \tilde{\psi} B \right)$$

and the kernel arising from the second is generated by [20, Section 3.3]

$$Q_s^2 Q_t^2 = \left(\psi\gamma_1 + \tilde{\psi}\gamma_2 \right)^2 \left(\psi\alpha_1 + \tilde{\psi}\alpha_2 \right)^2.$$

In terms of correlation functions, the result above implies that

$$\langle \mathcal{O} \rangle_{\vec{d}=0} \propto \langle \mathcal{O} Q_s Q_t \rangle_{\vec{d}=(0,1)}$$

which suggests that the operator product expansion (OPE) ring has the (partial) form

$$Q_s Q_t = q_2 \tag{20}$$

for some parameter q_2 .

Now, consider the sector of instanton degree $\vec{d} = (1, -n)$. In this sector, we need to take into account contributions from four-Fermi terms, something we have not needed to do previously. The LSM moduli space has homogeneous coordinates $u_{0,1}, v_{0,1}, t$ (where the u_i, v_i are the coefficients in the zero mode expansion of u, v and s does not contribute because it has no zero modes in this sector). These coordinates have weights

u_0	u_1	v_0	v_1	t
1	1	1	1	n
0	0	0	0	1

and the exceptional set is given by

$$\{u_0 = u_1 = v_0 = v_1 = 0, t = 0\}.$$

The induced bundle \mathcal{F} is given by

$$0 \longrightarrow \mathcal{F}^* \longrightarrow Z \xrightarrow{E'} W \otimes \mathcal{O} \longrightarrow 0,$$

where

$$Z = \mathcal{O}(-1, 0)^{\oplus 4} \oplus \mathcal{O}(-n, -1)$$

and the map E' , induced from E , is given by

$$E' = \begin{bmatrix} Ax_0 & Bx_0 \\ Ax_1 & Bx_1 \\ \alpha_1 t & \alpha_2 t \end{bmatrix},$$

where $x_i = [u_i, v_i]^T$. Furthermore, the second $U(1)$ effectively removes t from the moduli space, so the LSM moduli space is effectively \mathbf{P}^3 , and then² $Z = \mathcal{O}(-1)^{\oplus 4} \oplus \mathcal{O}$. In this example, \mathcal{F}_1 will be nonzero, as we will discuss momentarily, but first let us compute the cohomology ring structure in this instanton sector.

Proceeding based on previous experience, the kernel will have two components. One component will arise from $H^3(\Lambda^4 Z) \neq 0$. This kernel will be proportional to

$$Q_{K1}^2 = \left(\det \left(\psi A + \tilde{\psi} B \right) \right)^2.$$

The second component will arise from $H^0(Z) \neq 0$. This kernel will be proportional to

$$Q_t = \alpha_1 \psi + \alpha_2 \tilde{\psi}.$$

Now, let us compute \mathcal{F}_1 . We will find that four-Fermi terms will contribute, something that has not been true in previous cases. (As a result, the interpretation of the kernels computed above as kernels of correlation functions is more subtle than before — in some ways, this case is more closely parallel to the details of a single projective space and the kernels computed there.) Here

$$\mathcal{F}_1 = H^1(\mathbf{P}^1, \mathcal{O}(-n)) \otimes \mathcal{O}(0, 1) = \bigoplus_1^{n-1} \mathcal{O}(0, 1)$$

(for $n \geq 1$; we omit $n = 0$ as we have already studied $\mathbf{P}^1 \times \mathbf{P}^1$). If we describe the moduli space as \mathbf{P}^3 , then $\mathcal{F}_1 = \mathcal{O}(-n)^{\oplus n-1}$. (In previous cases, \mathcal{F}_1 vanished; we only mention it when it is nonzero.)⁹ Since \mathcal{F}_1 is nonzero (and of the same rank as the obstruction bundle, which in fact is identical), in each correlation function in this sector we need to insert

$$Q_s^{n-1} = \left(\psi \gamma_1 + \tilde{\psi} \gamma_2 \right)^{n-1}$$

(following Appendix A).

²The reader might ask why the last factor is \mathcal{O} instead of $\mathcal{O}(-n)$, since it arises from $\mathcal{O}(-n, -1)$. The answer is that the \mathbf{C}^\times action describing the \mathbf{P}^3 , must leave the t coordinate neutral. If we label the two \mathbf{C}^\times actions defining \mathbf{F}_n as λ, μ , then the \mathbf{C}^\times action defining $\mathcal{M} = \mathbf{P}^3$ is $\lambda - n\mu$, so that over that \mathcal{M} , t is a smooth section of \mathcal{O} and s is a smooth section of $\mathcal{O}(-n)$.

Now, let us find some relations between correlation functions. First, let us relate correlation functions in degree $(1, -n)$ to those in degree $(1, 0)$. In both degrees, Q_{K1}^2 partially generates the kernel, but in the former case, the rest arises from Q_t , whereas in the latter case, $Q_s Q_t^{n+1}$ is a generator, so to account for the difference, to map kernels to kernels, correlators in the degree $(1, 0)$ sector must be multiplied by $Q_s Q_t^{n+1}/Q_t = Q_s Q_t^n$. Furthermore, because in the degree $(1, -n)$ sector, four-Fermi terms add a factor of Q_s^{n-1} , we must also add that same factor to correlators in degree $(1, 0)$. Thus, we find that

$$\langle \mathcal{O} \rangle_{\vec{d}=(1,-n)} \propto \langle \mathcal{O} (Q_s Q_t^n) (Q_s^{n-1}) \rangle_{\vec{d}=(1,0)} = \langle \mathcal{O} (Q_s Q_t)^n \rangle_{\vec{d}=(1,0)}.$$

Note that this result is compatible with the earlier relation (19), namely

$$Q_s Q_t = q_2$$

for some constant q_2 ; furthermore, to achieve that compatibility required both matching kernels and also utilizing four-Fermi terms.

As one more consistency check, let us now work out the relation between correlation functions in degree $\vec{d} = 0$ and those in degree $\vec{d} = (1, -n)$. In the former case, the kernel is generated by Q_{K1} , $Q_s Q_t$, whereas in the latter case, the kernel is generated by Q_{K1}^2 , Q_t , so if we ignore four-Fermi terms, then to match kernels, correlation functions would be related by

$$\langle \mathcal{O} Q_s \rangle_{\vec{d}=0} \propto \langle \mathcal{O} Q_{K1} \rangle_{\vec{d}=(1,-n)}.$$

Because in degree $(1, -n)$ we also have four-Fermi terms, generating factors of Q_s^{n-1} , the correct relation between correlation functions is

$$\langle \mathcal{O} Q_s Q_s^{n-1} \rangle_{\vec{d}=0} \propto \langle \mathcal{O} Q_{K1} \rangle_{\vec{d}=(1,-n)}.$$

In terms of our previous relations, this suggests that

$$Q_{K1} = q_1 q_2^{-n} Q_s^n$$

which is indeed an algebraic consequence of (19) and (20). (Simply multiply both sides by either Q_s^n or Q_t^n and apply (20) to turn one into the other.) Again, note we need both kernels and four-Fermi terms to derive consistent relations.

This last example also illustrates a technical point regarding OPE computations that will arise in [20]. There, we will derive OPEs by giving relations

between correlation functions of the form

$$\langle \mathcal{O} \rangle_\beta \propto \langle \mathcal{O} R_{\beta, \beta'} \rangle_{\beta'}.$$

Except for the last case above, in the examples we have studied it has been possible to find an $R_{\beta, \beta'}$ putting relations in the form above. However, the last example illustrates that this cannot always be done. Technically, in [20] we deal with this issue through the introduction of “direct systems” to describe relations between correlation functions of different degrees.

Let us also take a moment to discuss the interpretation of the q ’s. In this text, we have been using them merely as placeholders for unspecified constants; in particular, classical limits do not necessarily correspond to the case that all $q_i \rightarrow 0$. To clarify this, let us consider the (2,2) limit of the relations we have been deriving. In this limit,

$$A = I, \quad B = 0, \quad \gamma_1 = 0, \quad \gamma_2 = 1, \quad \alpha_1 = n, \quad \alpha_2 = 1, \quad f_1 = f_2 = 0.$$

As a result

$$Q_{K1} = \psi^2, \quad Q_s = \tilde{\psi}, \quad Q_t = n\psi + \tilde{\psi}.$$

The classical cohomology ring of the Hirzebruch surface can be described by (toric) generators D_u, D_v, D_s, D_t in degree 2, obeying

$$D_u \sim D_v, \quad D_t \sim D_s + nD_v, \\ D_u^2 = 0, \quad D_s(nD_u + D_s) = 0.$$

If we identify $D_u = \psi, D_s = \tilde{\psi}$, then the relations (19) and (20), namely

$$Q_{K1} Q_t^n = q_1, \quad Q_s Q_t = q_2$$

become

$$D_u^2(nD_t + D_u) = q_1, \quad D_s(nD_u + D_s) = q_2$$

which clearly do not have the correct classical limit when $q_1 \rightarrow 0$. On the other hand, the equivalent relations

$$Q_{K1} = q' Q_s^n, \quad Q_s Q_t = q_2$$

(where $q' = q_1 q_2^{-n}$) become

$$D_u^2 = q' D_s^n, \quad D_s(nD_u + D_s) = q_2$$

which does reduce to the classical cohomology ring relations when $q' = q_1 q_2^{-n} \rightarrow 0$ and $q_2 \rightarrow 0$. This also illuminates the issue with the previous presentation — $q_1 \rightarrow 0, q_2 \rightarrow 0$ independently do not give the classical limit, one must also demand $q_1 q_2^{-n} \rightarrow 0$. Thus, we see that only certain presentations of the ring will give the classical cohomology ring on the (2,2) locus when all $q \rightarrow 0$. For other presentations, more complicated limits must be taken³. The particular presentation given in [20] does have the property that on the (2,2) locus, one recovers the correct classical limit as all $q \rightarrow 0$ independently. In this paper we shall not belabor this point.

Now, let us summarize. We have not described an exhaustive survey of all possibilities (see instead [20]), but based on the computations performed, it would seem that the OPE ring in this example is defined by

$$\begin{aligned} Q_{K1} Q_t^n &= q_1, \\ Q_s Q_t &= q_2 \end{aligned}$$

which are relations (19) and (20). We will see in Section 6 that this is a correct specialization of the general results of [20].

6 General result

6.1 Result

First, we shall outline the result from [20], and then compute it in examples.

Let $\{\rho_i\}$ denote the (one-dimensional) edges of the fan, i.e., the toric divisors, and let K_i denote “primitive collections” of edges, i.e., maximal collections of edges not contained in any single cone. (These collections define the Stanley–Reisner ideal, through the statement that the toric divisors do not all intersect.)

To each primitive collection K , we can associate a unique divisor class β_K , as follows. Let the vector generating the edge of the fan corresponding to ρ be denoted v_ρ , then for $K = \{\rho_1, \dots, \rho_k\}$, we can write

$$v_{\rho_1} + \dots + v_{\rho_k} = \sum_{\rho} c_{\rho} v_{\rho} \tag{21}$$

for some integers $c_{\rho} > 0$, with the sum on the right running over toric divisors not necessarily in K . By moving the right-hand side to the left, we can write

³We would like to thank I. Melnikov for illuminating discussions of this point.

this as

$$\sum_{\rho} a_{\rho} v_{\rho} = 0 \tag{22}$$

for some integers a_i . Then, it can be shown [24] that there is a unique curve class β_K such that $\beta_K \rho = a_{\rho}$ for all ρ .

Now, for each divisor class $c = [\rho]$, we define a $|c| \times |c|$ matrix A_c , where $|c|$ is the number of toric divisors linearly equivalent to ρ . The matrix A_c is given by the rows of the map E appearing in the definition of the deformation \mathcal{E}^* , the rows corresponding to representatives of c , and with nonlinear terms omitted. Define

$$Q_c = \det A_c.$$

The quantum sheaf cohomology ring is then given by polynomials in the elements of a basis for W , modulo the relations

$$\prod_{c \in [K]} Q_c = q^{\beta_K} \prod_{c \in [K^-]} Q_c^{-d_c^{\beta_K}} \tag{23}$$

for each primitive collection K , where $[K^-]$ denotes the set of linear equivalence classes of edges appearing in the right-hand side of (21) with nonzero coefficients c_{ρ} , and $d_c^{\beta_K} \equiv c \beta_K$. (Note that for $c \in [K^-]$, the exponent $-d_c^{\beta_K}$ is nonnegative.)

The formula above gives a canonical presentation of the quantum sheaf cohomology ring for each toric variety, dependent only upon the bundle and toric variety and not the details of any particular presentation such as \mathbf{C}^{\times} weights or $U(1)$ charges in a quotient.

Let us work through a few examples of this formalism, beginning with a projective space \mathbf{P}^n , as described in Section 3. Here, there are $n + 1$ toric divisors ρ_0, \dots, ρ_n . There is only one primitive collection

$$K = \{\rho_0, \dots, \rho_n\}$$

and for any fan, the vectors generating the edges obey

$$v_{\rho_0} + \dots + v_{\rho_n} = 0.$$

The unique divisor class β such that $\beta \rho = 1$ for all ρ is represented by any of the toric divisors ρ , since they are all linearly equivalent. All of the toric

divisors are linearly equivalent, and so there is one matrix $A_c = A\psi$, derived from the map E defining \mathcal{E}^* , and one $Q = \det A_c = \det(A\psi)$. The quantum sheaf cohomology ring is then $\mathbf{C}[\psi]$ modulo the relation

$$\det(A\psi) = q$$

matching what was found in Section 3, and for that matter matching the (2,2) locus (since on a single projective space, all toric Euler deformations return the tangent bundle itself).

A slightly more interesting example is $\mathbf{P}^1 \times \mathbf{P}^1$, as discussed in Section 4. Here, let $D_{x_0,1}, D_{\tilde{x}_0,1}$ denote the four toric divisors. (We are using here the nearly same notation for the toric divisors that we used for corresponding homogeneous coordinates in Section 4.) There are two primitive collections

$$K_1 = \{D_{x_0}, D_{x_1}\}, \quad K_2 = \{D_{\tilde{x}_0}, D_{\tilde{x}_1}\}.$$

In each primitive collection, the constituent divisors are all linearly equivalent to one another. Moreover

$$v_{x_0} + v_{x_1} = 0, \quad v_{\tilde{x}_0} + v_{\tilde{x}_1} = 0.$$

It is easy to check that β_1 is represented by $D_{\tilde{x}_0}, D_{\tilde{x}_1}$ and β_2 is represented by D_{x_0}, D_{x_1} , since $D_{x_i} D_{\tilde{x}_j} = 1$. Following the notation of Section 4, we find

$$\begin{aligned} A_1 &= \psi A + \tilde{\psi} B, \\ A_2 &= \psi C + \tilde{\psi} D \end{aligned}$$

and from (23) the quantum sheaf cohomology ring is $\mathbf{C}[\psi, \tilde{\psi}]$ modulo the relations

$$\begin{aligned} \det(\psi A + \tilde{\psi} B) &= q^{\beta_1}, \\ \det(\psi C + \tilde{\psi} D) &= q^{\beta_2} \end{aligned}$$

matching the results of Section 4.

Now, let us specialize the general result to Hirzebruch surfaces, and compare to the results we obtained previously. To that end, we describe the Hirzebruch surface classically with four (toric) divisors D_u, D_v, D_s, D_t ,

where

$$D_u = D_v, \quad D_t = D_s + nD_v$$

and

$$D_u D_v = 0 = D_s D_t.$$

There are two “primitive collections” of divisors, defined by the Stanley–Reisner ideal above

$$K_1 = \{D_u, D_v\}, \quad K_2 = \{D_s, D_t\}.$$

For the first primitive collection

$$v_u + v_v = nv_s$$

and the unique divisor class β_1 such that

$$[D_u]\beta_1 = 1 = [D_v]\beta_1, \quad [D_s]\beta_1 = -n, [D_t]\beta_1 = 0$$

is represented by D_s , i.e., $\beta_1 = [D_s]$. Similarly

$$v_s + v_t = 0$$

and the unique divisor class β_2 such that

$$[D_u]\beta_2 = 0 = [D_v]\beta_2, \quad [D_s]\beta_2 = 1 = [D_t]\beta_2$$

is represented by D_u, D_v , i.e., $\beta_{K_2} = [D_u] = [D_v]$.

Then, to each primitive collection K is associated a polynomial in the generators of W . In this case, these polynomials are

$$\prod_{c \in [K_1]} Q_c = Q_{K_1},$$

$$\prod_{c \in [K_2]} Q_c = Q_s Q_t.$$

In the expressions above, note that D_u and D_v are linearly equivalent, so there is only one linear equivalence class in $[K_1]$, but D_s and D_t are not

linearly equivalent, so there are two linear equivalence classes in $[K_2]$. Then, putting this together, the quantum sheaf cohomology relations in (23) are

$$\begin{aligned} \prod_{c \in [K_1]} Q_c &= Q_{K_1} = q^{\beta_1} Q_s^{-d_s^{\beta_1}} \\ &= q^{\beta_1} Q_s^n, \\ \prod_{c \in [K_2]} Q_c &= Q_s Q_t = q^{\beta_2}, \end{aligned}$$

where

$$d_\rho^{\beta_K} \equiv [D_\rho] \beta_K$$

(here, $d_s^{\beta_1} = [D_s] \beta_1 = -n$) and q^{β_1}, q^{β_2} are the two quantum parameters.

Note that by multiplying both sides by Q_t^n and using the second relation, we can turn these two relations into

$$\begin{aligned} Q_{K_1} Q_t^n &= q', \\ Q_s Q_t &= q^{\beta_2}, \end{aligned}$$

where $q' = q^{\beta_1} (q_2^\beta)^n$. It is this latter form in which the OPE rings for the Hirzebruch surface appear earlier in Section 5, where $q_1 = q', q_2 = q^{\beta_2}$.

6.2 Comparison to McOrist–Melnikov’s results

Let us now compare to the one-loop Coulomb branch results for the quantum sheaf cohomology ring given in [4].

Implicitly, the relations derived in all one-loop Coulomb branch computations are not the relations of the ring at any single large-radius limit of the GLSM, but rather live in a “localization” of the ring, in which operators have been inverted. Physically, this arises because the Coulomb branch computations take place in a regime where σ vevs are large, and so can be assumed nonzero and invertible; mathematically, this makes it possible for the one-loop Coulomb branch relations to be equally applicable to all large-radius phases. Thus, to compare the results of the last subsection, derived in a single large-radius phase, we must descend to a localization of the ring in which operator invertibility is allowed, and make the comparison in that localization. We will find that, after implicitly descending to that

localization, the results of the last subsection do indeed match the predictions of [4].

Partition the line bundle factors into collections $\{\mathcal{O}(\vec{q}_i)\}$ with matching \vec{q}_i . (We can think of this equivalently as partitioning the chiral superfields, indexed by i , into collections consisting of matching $U(1)$ charges \vec{q}_i .) Index such collections by α . (There is a one-to-one correspondence between such collections and linear equivalence classes of toric divisors.) Let

$$E_i : \mathcal{O}^{\oplus r} \longrightarrow \mathcal{O}(\vec{q}_i)$$

denote the maps in the short exact sequence defining \mathcal{E} . Define

$$A_{(\alpha)i}^{ja} \equiv \left. \frac{\partial}{\partial \phi_j} E_i^a \right|_{\phi \equiv 0}$$

for i, j in the collection α . For example, the tangent bundle of a toric variety is described by

$$E_i^a = Q_i^a \phi_i$$

hence

$$A_{(\alpha)i}^{ja} = \delta_i^j Q_{(\alpha)}^a,$$

where $\vec{q}_\alpha = (Q_\alpha^a)$ denotes the $U(1)$ charges of all fields in the collection α .

In this language, if we define V_α to be a vector space of the same dimension as the number of line bundles in the collection α (the number of chiral superfields with matching charges \vec{q}_α), and let $W = \mathbf{C}^r$, then we can describe the deformation of the tangent bundle as the cokernel

$$0 \longrightarrow W \otimes \mathcal{O} \longrightarrow \bigoplus_{\alpha} V_{\alpha} \otimes \mathcal{O}(\vec{q}_{\alpha}) \longrightarrow \mathcal{E} \longrightarrow 0.$$

Define

$$M_{(\alpha)i}^j = A_{(\alpha)i}^{ja} \psi_a.$$

This is the same matrix that was denoted A_c in the previous section, but we have adapted our notation to more closely resemble that of [4]. In this notation, the result of [20] is that the quantum sheaf cohomology ring relations

descend to

$$\prod_{\alpha} (\det M_{(\alpha)})^{Q_{\alpha}^a} = q_a \tag{24}$$

for each a , where $\vec{q}_{\alpha} = (Q_{\alpha}^a)$, and q_a is the quantum parameter, modulo inversion of operators.

The ring above is specified in terms of the $U(1)$ charges of the toric homogeneous coordinates, whereas in the previous section we gave a canonical representation that was independent of such choices. Specifically, the canonical representative was described in terms of a_{ρ} defined in (22). However, the charges Q_{α}^a are also defined as the kernel of a matrix formed from the v_{ρ} 's, as in equation (22); thus, the $a_{\rho} = D_{\rho}\beta$ defined there are precisely one set of charges. With that in mind, the quantum sheaf cohomology relations (23) can be written in the form

$$\prod_c Q_c^{D_c\beta} = q^{\beta}$$

for the β associated to each primitive collection, which is the same as

$$\prod_c (\det A_c)^{Q_c^{\beta}} = q^{\beta}$$

for $Q_c^{\beta} \equiv D_c\beta$. Thus, we see that the relations (23) specified in [20] really do descend to the relations (24) in a localization⁴ of the ring, written in a form closer to that of [4], for a particular choice of charges Q_{α}^a .

We should emphasize again that this result is independent of nonlinear deformations (meaning, terms in E_i^a nonlinear in ϕ 's), as conjectured in, e.g., [11, Section 3.5]. This result also nicely meshes with previous physics results. For example, Kreuzer *et al.* [13, Section A.3] conjectured that $A/2$ correlation functions should be independent of nonlinear deformations, based on the fact that the discriminant locus in GLSMs does not depend on such nonlinear deformations.

⁴We are implicitly performing this comparison in the localization mentioned earlier, as we have not specified whether the charges Q_{α}^a are positive or negative.

In the special case of linear deformations, i.e., when

$$E_i^a = \sum_j A_{(\alpha)_i}^{j_a} \phi_j$$

the result above specializes to the result of [4], computed with Coulomb branch techniques in GLSMs.

Now, let us compare to particular examples discussed earlier in this paper.

In the case of deformations of the tangent bundle of $\mathbf{P}^1 \times \mathbf{P}^1$ discussed in Section 4.3, it is straightforward to check that there are two $M_{(\alpha)}$, given by

$$\begin{aligned} M_{(1)} &= \psi_1 A + \psi_2 B, \\ M_{(2)} &= \psi_1 C + \psi_2 D \end{aligned}$$

and so we have the relations

$$\det M_{(1)} = q_1, \quad \det M_{(2)} = q_2$$

which matches our previous computation.

Next, let us describe the example of a Hirzebruch surface \mathbf{F}_n . Consider a fan with edges $(1, 0)$, $(0, 1)$, $(-1, n)$, $(0, -1)$, defined by the charges $(1, 0)$, $(0, 1)$, $(1, 0)$, $(n, 1)$ and homogeneous coordinates u, s, v, t , respectively, as in figure 1. For a deformation of the tangent bundle of \mathbf{F}_n as defined in (1), we compute

$$\begin{aligned} M_{(1)} &= A\psi_1 + B\psi_2, \\ M_{(2)} &= \gamma_1\psi_1 + \gamma_2\psi_2, \\ M_{(3)} &= \alpha_1\psi_1 + \alpha_2\psi_2 \end{aligned}$$

and so we have the quantum sheaf cohomology relations

$$\begin{aligned} (\det M_{(1)}) (M_{(3)})^n &= q_1, \\ (M_{(2)}) (M_{(3)}) &= q_2 \end{aligned}$$

which are precisely (19) and (20) computed earlier, identifying

$$Q_{K1} = \det M_{(1)}, \quad Q_s = M_{(2)}, \quad Q_t = M_{(3)}.$$

In the special case that $\mathcal{E} = TX$, the ring above reduces to

$$\prod_i \left(\sum_b Q_i^b \psi_b \right)^{Q_i^a} = q_a$$

or equivalently

$$\begin{aligned} \psi_1^2 (n\psi_1 + \psi_2)^n &= q_1, \\ \psi_2 (n\psi_1 + \psi_2) &= q_2 \end{aligned}$$

which is a standard result in (2,2) GLSMs [23, equation (3.44)]. If we identify the toric divisors D_i as

$$D_i = \sum_a Q_i^a \psi_a$$

(as a consistency check, note that since $\sum_i Q_i^a \vec{v}_i = 0$, it is necessarily the case that

$$\sum_i \langle m, \vec{v}_i \rangle D_i = 0$$

hence the description above encodes the linear relations on the Chow ring) then the GLSM ring can be written as

$$\prod_i D_i^{Q_i^a} = q_a$$

or equivalently

$$\begin{aligned} D_u D_v D_t^n &= q_1, \\ D_s D_t &= q_2, \end{aligned}$$

where

$$D_u \sim D_v, \quad D_t \sim D_s + nD_v.$$

There is an issue with classical limits, previously noted in Section 5. Let us outline the analysis here, in the language of GLSMs. The classical cohomology ring of \mathbf{F}_n can be described by the relations

$$D_u^2 = 0, \quad D_s^2 = -nD_u D_s$$

and if we identify $D_u = \psi_1$, $D_s = \psi_2$, then we almost recover this in the limit $q_a \rightarrow 0$, except for an extra factor of D_t^n modifying the relation $D_u^2 = 0$. In

order to make the relation with the classical cohomology ring more clear, we should work in a different basis, one in which the fields have charges $(1, 0)$, $(1, 0)$, $(0, 1)$, $(-n, 1)$. (Note this is achieved by an $SL(2, \mathbf{Z})$ transformation.) In this basis, the quantum cohomology ring becomes

$$\psi_1^2 = q_1 (-n\psi_1 + \psi_2)^n, \quad \psi_2 (-n\psi_1 + \psi_2) = q_2$$

and so when we set $q_a \rightarrow 0$, and identify $D_u = \psi_1$, $D_s = -\psi_2$, we recover the classical cohomology ring without extraneous factors. (Alternatively, the canonical presentation of the previous section avoids this problem.) More invariantly, to cleanly recover the classical cohomology relations from the GLSM relations, one wants to work in a basis such that the smooth phase of the GLSM corresponds to the positive orthant of the secondary fan; this is a property of the canonical presentation of the previous section.

7 Conclusions

In this paper we have outlined the mathematical computation of quantum sheaf cohomology rings for deformations of tangent bundles of toric varieties, emphasizing physics aspects of the computation. Our new methods allow for much more efficient mathematical computations than possible previously. We have also seen in examples that in these cases (toric varieties, deformations of the tangent bundle), quantum sheaf cohomology is independent of nonlinear deformations, as conjectured elsewhere (see, e.g., [11, 13]). Rigorous general proofs will appear in [20].

Extensions of the results of this paper to Grassmannians and flag manifolds are under discussion [25]. Extensions to hypersurfaces would also be extremely useful. In this paper and [20] we compute kernels of correlation functions in order to compute operator products; it would also be interesting to work out complete expressions for the correlation functions themselves.

8 Acknowledgments

We would like to thank M. Ballard, J. McOrist and I. Melnikov for useful conversations. This work was presented in part at the conferences “String-Math 2011” at the University of Pennsylvania, and “(0,2) mirror symmetry and heterotic topological strings” at the Erwin Schrödinger Institute in Vienna, Austria, and we thank both institutions for hospitality while part of this work was completed. R.D. was partially supported by NSF grants DMS-0908487, DMS-0636606. J.G. was partially supported by NSF grant

DMS-0636606. S.K. was partially supported by NSF grant DMS-0555678. E.S. was partially supported by NSF grants DMS-0705381, PHY-0755614.

Appendix A. GLSM derivation of four-Fermi terms

In this section we will outline how four-Fermi effects arise in GLSMs, and use this to derive the ansatz for their effects in (0,2) theories used earlier in Sections 5 and 6. Furthermore, we will see explicitly that nonlinear deformations cannot contribute to the four-Fermi terms, at least in the GLSM.

First, let us consider ordinary (2,2) supersymmetric GLSMs for toric varieties, as in [23, 26]. As discussed in [27], correlation functions in the A-twisted theory are correlation functions of products of $\bar{\sigma}$'s. The GLSM itself does not contain any four-Fermi terms, but the effects of four-Fermi terms in the low-energy NLSM are duplicated by Yukawa couplings in the GLSM of the form

$$\sum_{i,a} Q_i^a \bar{\sigma}_a \psi_z^{\bar{i}} \psi_{\bar{z}}^i.$$

Four-Fermi terms in a NLSM must be invoked whenever $\psi_z^{\bar{i}}$, $\psi_{\bar{z}}^i$ have zero modes. In the present case, when such fields have zero modes, to absorb them one must use the Yukawa couplings above, which will then be responsible for a factor of

$$\prod_i \left(\sum_a Q_i^a \bar{\sigma}_a \right)^{n_i},$$

where

$$n_i = h^1 \left(\mathbf{P}^1, \mathcal{O} \left(\vec{Q}_i \vec{d} \right) \right),$$

where \vec{Q}_i has components Q_i^a , and \vec{d} defines the degree of the instanton sector. (Strictly speaking, four-Fermi interactions contribute integrals over of the worldsheet of such factors; however, in the A model, correlators are independent of position, so such integrals merely contribute factors of the worldsheet area, which are cancelled out by corresponding factors in the path integral. See [27] for details.) This precisely duplicates the contribution described in [1, Section 6.2.2]; [23, equation (3.69)].

In an A/2 theory describing a toric variety with a deformation of the tangent bundle, the results above are modified slightly. There, the pertinent

Yukawa couplings are of the form [14, 26]

$$\sum_{i,j} \psi_{\bar{z}}^{\bar{i}} \psi_{\bar{z}}^j \partial_{\bar{i}} \bar{E}^{\bar{j}}.$$

In terms of correlation functions, this means we insert

$$\prod_c (\det A_c)^{n_c},$$

where c runs over classes of toric divisors with the same GLSM charge (i.e., linear equivalence classes)

$$n_c = h^1(\mathbf{P}^1, \mathcal{O}(\vec{Q}_c \vec{d})),$$

where \vec{Q}_c is the charge of the homogeneous coordinates in class c , and the matrix

$$A_c = (\partial_i E^j)$$

with i, j running in the same class c . (Strictly speaking, again, four-Fermi terms contribute integrals. In the A/2 model, a priori worldsheet correlators are holomorphic functions of position, but as argued in [7] for CFT's and [28] for GLSMs, in a neighbourhood of the (2,2) locus, A/2 correlators are actually independent of position, and so again the integrals merely contribute factors of worldsheet area.)

Let us work out an example in detail to illustrate what this means. Consider the example of a Hirzebruch surface \mathbf{F}_n , as in Section 2. In an example there, the deformation \mathcal{E} of the tangent bundle is described by

$$0 \longrightarrow \mathcal{E}^* \longrightarrow \mathcal{O}(-1, 0)^2 \oplus \mathcal{O}(0, -1) \oplus \mathcal{O}(-n, -1) \xrightarrow{E} W \otimes \mathcal{O} \longrightarrow 0,$$

where

$$E = \begin{bmatrix} Ax & Bx \\ \gamma_1 s & \gamma_2 s \\ \alpha_1 t + s f_1(u, v) & \alpha_2 t + s f_2(u, v) \end{bmatrix}$$

with

$$x \equiv \begin{bmatrix} u \\ v \end{bmatrix}$$

A, B are constant 2×2 matrices, $\gamma_1, \gamma_2, \alpha_1, \alpha_2$ are constants, and $f_{1,2}(u, v)$ are homogeneous polynomials of degree n .

In the case above

$$\begin{aligned}
 E^u &= (A_{11}u + A_{12}v) \sigma_1 + (B_{11}u + B_{12}v) \sigma_2, \\
 E^v &= (A_{21}u + A_{22}v) \sigma_1 + (B_{21}u + B_{22}v) \sigma_2, \\
 E^s &= \gamma_1 s \sigma_1 + \gamma_2 s \sigma_2, \\
 E^t &= (\alpha_1 t + s f_1(u, v)) \sigma_1 + (\alpha_2 t + s f_2(u, v)) \sigma_2
 \end{aligned}$$

and the pertinent Yukawa couplings are

$$\begin{aligned}
 &\psi_z^{\bar{u}} \psi_{\bar{z}}^u \partial_{\bar{u}} \bar{E}^{\bar{u}} + \psi_z^{\bar{v}} \psi_{\bar{z}}^v \partial_{\bar{v}} \bar{E}^{\bar{v}} + \psi_z^{\bar{u}} \psi_{\bar{z}}^v \partial_{\bar{u}} \bar{E}^{\bar{v}} + \psi_z^{\bar{v}} \psi_{\bar{z}}^u \partial_{\bar{v}} \bar{E}^{\bar{u}} \\
 &+ \psi_z^{\bar{s}} \psi_{\bar{z}}^s \partial_{\bar{s}} \bar{E}^{\bar{s}} + \psi_z^{\bar{t}} \psi_{\bar{z}}^t \partial_{\bar{t}} \bar{E}^{\bar{t}} + \psi_z^{\bar{s}} \psi_{\bar{z}}^t \partial_{\bar{s}} \bar{E}^{\bar{t}} + \psi_z^{\bar{t}} \psi_{\bar{z}}^s \partial_{\bar{t}} \bar{E}^{\bar{s}} + \psi_z^{\bar{u}} \psi_{\bar{z}}^t \partial_{\bar{u}} \bar{E}^{\bar{t}} + \psi_z^{\bar{v}} \psi_{\bar{z}}^t \partial_{\bar{v}} \bar{E}^{\bar{t}}
 \end{aligned}$$

(and their complex conjugates). In particular, the last three terms

$$\psi_z^{\bar{s}} \psi_{\bar{z}}^t \partial_{\bar{s}} \bar{E}^{\bar{t}} + \psi_z^{\bar{u}} \psi_{\bar{z}}^t \partial_{\bar{u}} \bar{E}^{\bar{t}} + \psi_z^{\bar{v}} \psi_{\bar{z}}^t \partial_{\bar{v}} \bar{E}^{\bar{t}}$$

completely encode the nonlinear terms $s f_i(u, v)$ in E — those nonlinear terms do not enter into any of the other Yukawa couplings above.

Because the couplings containing the nonlinear terms are not paired symmetrically, because $\partial_t E^{u,v,s} = 0$, those nonlinear terms will not affect the four-Fermi contribution to correlation functions. To see this, first note that when we integrate over $\psi_z, \psi_{\bar{z}}$ zero modes, we will take a determinant of Yukawa couplings (or rather, what those couplings induce over the instanton moduli space). Since determinants are antisymmetric, and $\partial_t E^{u,v} = 0$, it must be the case that terms involving $\partial_{u,v} E^t$ cannot contribute. (At a more elementary level, this is saying that since we can evaluate determinants along either rows or columns, if we choose to evaluate along a column with only one nonzero entry, then nonzero entries in the transpose row cannot contribute.)

On any compact toric variety, the same will be true more generally. The point is that for the argument to fail, we need for it to be possible to build gauge-invariant combinations of the homogeneous coordinates. However, that can be done if and only if the toric variety is noncompact (in which case, the vev of such gauge-invariant combinations corresponds to noncompact directions).

Let us work through the details in a particular case. Consider the sector of maps of degree $\vec{d} = (1, -n)$, which corresponds to maps mapping into the exceptional curve E with degree 1. This case was discussed earlier in this

paper in detail; in effect, we are merely giving a more detailed derivation of a result used there. In this case, it is straightforward to compute

$$0 \longrightarrow \mathcal{F}^* \longrightarrow \mathcal{O}(-1)^4 \oplus \mathcal{O} \xrightarrow{E'} W \otimes \mathcal{O} \longrightarrow 0$$

over $\mathcal{M} \cong \mathbf{P}^3$, with $\mathcal{F}_1 \cong \mathcal{O}(-n)^{n-1}$. In this case, there are only $\psi_{\bar{z}}^\mu$ zero modes for $\mu = s$, hence the only pertinent Yukawa coupling is the induced coupling⁵

$$\sum_i \psi_{z,i}^{\bar{s}} \psi_{\bar{z},i}^s (\gamma_1^* \bar{\sigma}_1 + \gamma_2^* \bar{\sigma}_2).$$

Integrating over the $\psi_z, \psi_{\bar{z}}$ zero modes gives a factor of

$$(\gamma_1^* \bar{\sigma}_1 + \gamma_2^* \bar{\sigma}_2)^{n-1}.$$

References

- [1] S. Katz and E. Sharpe, *Notes on certain (0,2) correlation functions*, Commun. Math. Phys. **262** (2006), 611–644; arXiv: hep-th/0406226.
- [2] A. Adams, A. Basu and S. Sethi, *(0,2) duality*, Adv. Theor. Math. Phys. **7** (2004), 865–950; arXiv: hep-th/0309226.
- [3] J. Guffin and S. Katz, *Deformed quantum cohomology and (0,2) mirror symmetry*, J. High Energy Phys. **1008** (2010), 109; arXiv: 0710.2354
- [4] J. McOrist and I. Melnikov, *Half-twisted correlators from the Coulomb branch*, J. High Energy Phys. **0804** (2008), 071; arXiv: 0712.3272.
- [5] E. Sharpe, *Notes on correlation functions in (0,2) theories*, in ‘String Geometry’ (Snowbird, 2004), eds. Katrin Becker, Melanie Becker, Aaron Bertram, Paul Green, and Benjamin McKay, American Mathematical Society, Providence, RI, 2004, 93–104; arXiv: hep-th/0502064.
- [6] E. Sharpe, *Notes on certain other (0,2) correlation functions*, Adv. Theor. Math. Phys. **13** (2009), 33–70; arXiv: hep-th/0605005.
- [7] A. Adams, J. Distler and M. Ernebjerg, *Topological heterotic rings*, Adv. Theor. Math. Phys. **10** (2006), 657–682; arXiv: hep-th/0506263.

⁵In general, we must compute what each coupling $\partial_s E^j$ induces over the moduli space. In the present case, we need only consider $\partial_s E^s$, which is independent of u, v, s and t , and hence takes the same form as it does classically.

- [8] M.-C. Tan, *Two-dimensional twisted sigma models and the theory of chiral differential operators*, Adv. Theor. Math. Phys. **10** (2006), 759–851; [arXiv:hep-th/0604179](#).
- [9] M.-C. Tan, *Two-dimensional twisted sigma models, the mirror Chiral de Rham complex, and twisted generalized mirror symmetry*, J. High Energy Phys. **0707** (2007), 013; [arXiv:0705.0790](#).
- [10] I. Melnikov and S. Sethi, *Half-twisted $(0,2)$ Landau–Ginzburg models*, J. High Energy Phys. **0803** (2008), 040; [arXiv:0712.1058](#).
- [11] J. McOrist and I. Melnikov, *Summing the instantons in half-twisted linear sigma models*, J. High Energy Phys. **0902** (2009), 026; [arXiv:0810.0012](#).
- [12] I. Melnikov, *$(0,2)$ Landau–Ginzburg models and residues*, J. High Energy Phys. **0909** (2009), 118; [arXiv:0902.3908](#).
- [13] M. Kreuzer, J. McOrist, I. Melnikov and R. Plesser, *$(0,2)$ deformations of linear sigma models*, [arXiv:1001.2104](#).
- [14] J. Guffin and E. Sharpe, *A-twisted heterotic Landau–Ginzburg models*, J. Geom. Phys. **59** (2009), 1581–1596; [arXiv:0801.3955](#).
- [15] I. Melnikov and R. Plesser, *A $(0,2)$ mirror map*, JHEP, J. High Energy Phys. **1102** (2011), 001; [arXiv:1003.1303](#).
- [16] P. Aspinwall, I. Melnikov and R. Plesser, *$(0,2)$ elephants*, [arXiv:1008.2156](#).
- [17] J. McOrist, *The revival of $(0,2)$ linear sigma models*, Int. J. Mod. Phys. A **26** (2011), 1–41; [arXiv:1010.4667](#).
- [18] J. Guffin, *Quantum sheaf cohomology, a precis*, [arXiv:1101.1305](#).
- [19] J. McOrist and I. Melnikov, *Old issues and linear sigma models*, [arXiv:1103.1322](#).
- [20] R. Donagi, J. Guffin, S. Katz and E. Sharpe, *A mathematical theory of quantum sheaf cohomology*, [arXiv:1110.3751](#).
- [21] E. Witten, *Mirror manifolds and topological field theory*, in ‘Mirror Symmetry I’, ed. S.-T. Yau, American Mathematical Society, Providence, Rhode Island, 1998, 121–160; [arXiv:hep-th/9112056](#).
- [22] D. Cox, J. Little and H. Schenk, *Toric varieties*, unpublished manuscript, available at: www.cs.amherst.edu/~dac/toric.html
- [23] D. Morrison and R. Plesser, *Summing the instantons: quantum cohomology and mirror symmetry in toric varieties*, Nucl. Phys. **B440** (1995), 279–354; [arXiv:hep-th/9412236](#).
- [24] V. Batyrev, *Quantum cohomology rings of toric manifolds*, Astérisque **218** (1993), 9–34.
- [25] R. Donagi, J. Guffin, S. Katz and E. Sharpe, work in progress.

- [26] E. Witten, *Phases of $N=2$ theories in two dimensions*, Nucl. Phys. **B403** (1993), 159–222; [arXiv:hep-th/9301042](#).
- [27] J. Guffin and E. Sharpe, *A-twisted Landau–Ginzburg models*, J. Geom. Phys. **59** (2009), 1547–1580; [arXiv:0801.3836](#).
- [28] I. Melnikov, private communication.

